1. In this paper we study the character of the convergence in probability of the Fourier series of stationary Gaussian processes. To simplify the notation we shall consider the expansion in a Fourier series on the interval \([-\pi, \pi]\). Thus, if \(f \in L^1(-\pi, \pi)\), then the partial sums of the Fourier series of the function \(f\) are defined by the convolutions \(D_n \star f\), \(n \geq 1\), where

\[
(D_n \star f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) D_n(t-u) \, du, \quad t \in \mathbb{R},
\]

and \(D_n, n \geq 1\), are the Dirichlet kernels:

\[
D_n(u) = \frac{\sin((n + 1/2)u)}{\sin u/2}, \quad u \in \mathbb{R}.
\]

It is well known in the theory of trigonometric Fourier series \([1]\) that the continuity of the function \(f\) is not sufficient for the uniform convergence of its Fourier series on the interval \([-a, a] \subset (-\pi, \pi)\). The situation is unchanged if instead of a determinate function \(f\) one considers a general (and even Gaussian) centered stochastic process, and convergence in the uniform norm is replaced by convergence in the uniform norm in probability. However, if \(f = f(t), t \in [-\pi, \pi]\) is the restriction to the interval \([-\pi, \pi]\) of a real stationary centered process and the process \(f\) is continuous almost surely (a.s.), then \([2]\) for any \([-a, a] \subset (-\pi, \pi)\)

\[
\|f - D_n \star f\|_{\infty, [-a, a]} \to 0,
\]

where \(\|\cdot\|_\infty\) is the uniform norm on \([-a, a]\). It follows in particular from this that from any sequence of partial sums of the Fourier series of the process \(f\), one can extract a subsequence which converges to \(f\) uniformly a.s. on any interval \([-a, a] \subset (-\pi, \pi)\). It will be shown below that an analogous assertion is also valid for convergences in probability of the Fourier series of the process \(f\) in norms generated by a large class of moduli of continuity.

Let \(\sigma(x), x > 0\) be a modulus of continuity, i.e., a nonnegative continuous function \(\sigma\), such that \(\sigma(0) = 0, \sigma(x) \leq \sigma(x + y) \leq \sigma(x) + \sigma(y)\) for \(x, y \geq 0\). In addition we shall assume that

\[
\lim_{x \to 0} \frac{x}{\sigma(x)} = 0. \tag{1}
\]

The simplest example of such moduli of continuity is the functions \(x^\alpha\) for \(\alpha \in (0, 1)\). Let \(C[a, b]\) be the space of real continuous functions on the interval \([a, b]\), and \(H^\sigma[a, b]\) be the space of the functions \(\varphi\) from \(C[a, b]\), for which

\[
\|\varphi\|_{[a, b]} = \sup_{t \in [a, b]} |\varphi(t)| + \sup_{t \in [a, b]} \frac{|\varphi(t) - \varphi(s)|}{\sigma(t - s)} \tag{2}
\]

is finite. With respect to the norm \((2)\), the space \(H^\sigma[a, b]\) is an inseparable Banach space. However its subspace \(H^\sigma[a, b]\), consisting of functions \(\varphi\), satisfying the condition \(\sup_{t - s < \delta} |\varphi(t) - \varphi(s)| = o(\sigma(\delta))\), will be a separable Banach space with respect to the norm \((2)\). In what follows...
we shall consider the restrictions of functions to different intervals. Let us agree that for a function \( \varphi \), defined on the parameter set \( T \), the notation \( \varphi \in \Phi (T_i) \), where \( \Phi (T_i) \) is a class of functions defined on \( T_i \subset T \), means that the restriction of \( \varphi \) to \( T_i \) belongs to the class \( \Phi (T_i) \). We consider stationary processes to be real and defined on the whole real line \( R \). All random elements are defined on a base probability space \((\Omega, F, P)\).

**Theorem 1.** Let \( \xi \) be a stationary centered Gaussian stochastic process. If \( \xi \in \mathcal{H}_0^0[-\pi, \pi] \) a.s., then for any \( a \in (0, \pi) \)

\[
\| \xi - D_n \xi \|_{[-a, a]} \xrightarrow{P} 0.
\]

Since the process \( \xi \) is not generally periodic, in Theorem 1 the interval \([-a, a]\) cannot be set equal to \([-\pi, \pi]\). Now if \( \xi \) is periodic and \( 2\pi \) is equal to the period or to a multiple of it, then the process \( \xi \) has discrete spectrum and its Fourier series will be a Fourier series with independent centered Gaussian coefficients. For series of independent random elements in separable Banach spaces, convergence in probability and convergence a.s. are equivalent. Hence in the periodic case in Theorem 1, convergence in probability can be replaced by convergence a.s., and as a one can choose any positive number. This fact accords completely with the known results [3, 4].

2. The proof of Theorem 1 is based on a number of auxiliary assertions. First of all we dwell on certain properties of the Dirichlet kernels. If \( f \in L_1[-2\pi, 2\pi] \), then along with the convolutions \( D_n f, n \gg 1 \), one can consider the convolutions \( f \ast D_n, n \gg 1 \), where

\[
(f \ast D_n)(t) = \frac{1}{n} \int_{-\pi}^{\pi} f(t-u) D_n(u) \, du, \quad t \in R.
\]

If the function \( f \) is periodic with period \( 2\pi \), then for all \( f \in [-\pi, \pi] \), \((D_n f)(t) = (f \ast D_n)(t) \). In general there is no such coincidence. However for \( n \) tending to infinity, the convolutions \( f \ast D_n \) and \( D_n \ast f \) approach one another inside the interval \([-\pi, \pi]\), where the character of the approach is identical with the degree of smoothness of the function \( f \).

**Lemma 1.** If \( f \in \mathcal{H}_0^0[-2\pi, 2\pi] \), then for any \( a \in (0, \pi) \),

\[
\| f \ast D_n - D_n \ast f \|_{[-a, a]} \xrightarrow{P} 0.
\]

The proof of this possibly familiar fact is based on rather involved but completely standard arguments, connected with application of the uniform version of the Riemann-Lebesgue theorem. We note that precisely in the proof of Lemma 1 one uses the restriction (1) on the modulus of continuity \( \sigma \).

**Lemma 2.** For any \( b > 0 \),

\[
\sup_{n \gg 1} \sup_{t \in [a, b]} \left| \int_{-b}^{b} \cos \lambda u D_n(u) \, du \right| < \infty.
\]

The proof of Lemma 2 has a purely technical character and involves no serious difficulties.

We recall that a family of probability measures \( (P_n, n \gg 1) \) in a Banach space \( E \) is said to be dense, if for any \( \varepsilon > 0 \) one can find a compactum \( K_\varepsilon \subset E \), such that \( P_n(K_\varepsilon) \geq 1 - \varepsilon \) for all \( n \gg 1 \).

**Lemma 3.** Let \( X_n = (X_n(t), t \in [a, b]) \), \( n \gg 0 \), be random elements in the space \( \mathcal{H}_0^0[a, b] \); \( P_n \) be the distribution of the random element \( X_n, n \gg 1 \). Then if

\[
\| X_0 - X_n \|_{L_1[a, b]} \xrightarrow{P} 0
\]

and the family of distributions \( (P_n, n \gg 1) \) is dense in the space \( \mathcal{H}_0^0[a, b] \), then

\[
\| X_0 - X_n \|_{L_1[a, b]} \xrightarrow{P} 0.
\]

Lemma 3 follows directly from Yu. V. Prokhorov's test for weak convergence of a sequence of distributions in Banach spaces (cf. also [3, p. 51]).

We apply Lemma 1 to the convolutions of the process \( \xi \) with a sequence of \( \delta \)-shaped kernels. Let \( b \gg a > 0 \) and \( (g_n, n \gg 1) \subset L_1[-b, b] \) be a sequence such that for any function \( f \in L_1[-2b, 2b] \) the