We construct two-dimensional splines and give two versions of an estimate of the deviation of splines from approximated functions. We compare approximations by a planar broken line and by a harmonic spline. We also substantiate the advisability of introduction of the notion of harmonic splines in mathematics.

Denote by $\Omega = \{0 < x < a, 0 < y < b\}$ a rectangular region in a two-dimensional Euclidean space. We denote the boundary of this region by $\partial \Omega$ and its closure by $\overline{\Omega}$. Similar notation is also used for other rectangular regions.

By using a net formed of lines parallel to the coordinate axes, we divide $\Omega$ into regions $\Omega_{ij} = \{x_i < x < x_{i+1}, y_j < y < y_{j+1}\}, \ i = 0, 1, \ldots, m - 1, \ j = 0, 1, \ldots, n - 1, \ x_0 = 0, \ x_m = a, \ y_0 = 0, \ y_n = b$. Denote the set of net nodes by $\Omega^* = \cup \Omega_{ij}$.

Assume that a function $u(x, y)$ is $2p$ times continuously differentiable with respect to $x$ and $y$ in the regions $\Omega_{ij}$ and the values of its derivatives $\partial^k u(x, y)/\partial \bar{n}^k, \ k = 0, 1, \ldots, p - 1$, are given on the contours $\partial \Omega_{ij}$; here, $\bar{n}$ is the direction of the inner normal to $\partial \Omega_{ij}$.

**Definition.** We say that a piecewise polyharmonic function $S_u(\Delta^p; x, y)$ is a polyharmonic spline of order $p$ that approximates a function $u(x, y)$ if it satisfies the equation

$$\Delta^p S_u(\Delta^p; x, y) = 0, \ (x, y) \in \Omega_{ij}$$

($\Delta$ is the Laplace operator) and the boundary conditions

$$\frac{\partial^k S_u(\Delta^p; x, y)}{\partial \bar{n}^k} \bigg|_{(x, y) \in \Omega^*} = \frac{\partial^k u(x, y)}{\partial \bar{n}^k} \bigg|_{(x, y) \in \Omega^*}, \ k = 0, 1, \ldots, p - 1.$$  

For $p = 1$, the polyharmonic spline $S_u(\Delta^1; x, y)$ is called harmonic and denoted by $S_u(\Delta; x, y)$.

A harmonic spline can be represented as the sum

$$S_u(\Delta; x, y) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij}(x, y),$$

where $u_{ij}(x, y)$ are functions satisfying the equation

$$\Delta u_{ij}(x, y) = 0, \ (x, y) \in \Omega_{ij},$$

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and the Dirichlet boundary conditions

\[
\lim_{(x, y) \to \Omega_{ij}} u_{ij}(x, y) = u(x, y) |_{(x, y) \in \Omega_{ij}}.
\]  

(5)

We also assume that

\[
u_{ij}(x, y) |_{(x, y) \in \Omega_{ij}} = 0.
\]  

(6)

Note that the limit transition in the boundary conditions (5) is carried out to exclude the possibility of superposition of the values of two functions \( u_{ij}(x, y) \) on internal lines of the net \( \hat{\Omega}^* \) that may follow from equality (6).

A solution of the boundary-value problem (4), (5) can be written in terms of a double-layer potential as follows:

\[
u_{ij}(x, y) = \int_{\Omega_{ij}} \frac{\partial G_{ij}(x, y; \xi, \eta)}{\partial n} u(\xi, \eta) d\Omega_{ij},
\]  

(7)

where \( G_{ij}(x, y; \xi, \eta) \) is Green's function satisfying the equation

\[
\Delta G_{ij}(x, y; \xi, \eta) = -\delta(x - \xi; y - \eta)
\]  

(8)

(\( \delta(x - \xi; y - \eta) \) is the Dirac delta function concentrated at the point \( (\xi, \eta) \) [1]) and the boundary conditions

\[
u_{ij}(x, y; \xi, \eta) |_{(x, y) \in \Omega_{ij}} = 0.
\]  

(9)

There are many known representations of Green's function \( G_{ij}(x, y; \xi, \eta) \). We use the following two of them [1]:

\[
G_{1,ij}(x, y; \xi, \eta) = -\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k \sinh(\alpha_{ki} h_j)} \left[ J(y - \eta) \sinh(\alpha_{ki}(y - y_{j+1})) \sinh(\alpha_{ki}(\eta - y_j)) + J(\eta - y) \sinh(\alpha_{ki}(y - y_j)) \sinh(\alpha_{ki}(\eta - y_{j+1})) \right] \sin(\alpha_{ki}(x - x_i)) \sin(\alpha_{ki}(\xi - x_i)),
\]  

(10)

\[
G_{2,ij}(x, y; \xi, \eta) = -\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k \sinh(\beta_{kj} l_i)} \left[ J(x - \xi) \sinh(\beta_{kj}(x - x_{i+1})) \sinh(\beta_{kj}(\xi - x_i)) + J(\xi - x) \sinh(\beta_{kj}(x - x_i)) \sinh(\beta_{kj}(\xi - x_{i+1})) \right] \sin(\beta_{kj}(y - y_i)) \sin(\beta_{kj}(\eta - y_j)),
\]  

(11)

where \( l_i = x_{i+1} - x_i \), \( h_j = y_{j+1} - y_j \), \( \alpha_{ki} = k\pi/l_i \), \( \beta_{kj} = k\pi/h_j \), and \( J(x - \xi) \) is the Heaviside function defined by the equalities

\[
J(x - \xi) = \begin{cases} 
1, & \xi \leq x; \\
0, & \xi > x. 
\end{cases}
\]