A METHOD FOR SOLUTION OF NONLINEAR
ORDINARY DIFFERENTIAL EQUATIONS

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We present a method for the solution of nonlinear second-order differential equations by using a system
of Fredholm equations of the second kind.

The aim of the present paper is to develop an efficient method for the numerical solution of the initial-value and
boundary-value problems for nonlinear ordinary differential equations on bounded intervals. Let us illustrate the
basic ideas by the scheme of solution of the Emden equation for polytropic gas

\[ u'' + \frac{2}{x} u' + \beta^2 u^\nu = 0, \quad x \in [0, \pi], \]

where \( 0 < \nu < 5 \) and \( \beta = \frac{x_0}{\pi} \), with the conditions \( u(0) = 1 \) and \( u'(0) = 0 \).

The value of \( \beta \) depends on \( \nu \); the cases \( \nu = 0 \) and \( \nu = 1 \) are trivial [1, pp. 347–369]. In this paper, we
present results of qualitative analysis of the behavior of solution. Similar problems for the more general Emden-
Fowler equations were studied in [2, Chap. 7], where the impossibility of finding a solution in the explicit form is
stated. The contemporary survey of this theory is given in [3, Chap. 5].

We consider the problem of finding a constructive algorithm for the calculation of the function \( u > 0 \) and the
parameter \( \beta \) under the additional condition \( u(\pi) = 0 \).

1. Reduction to Integral Equation

Denote

\[ a = -\varepsilon \beta^2 u^{\nu - 1}(x), \]

where \( \varepsilon \) is a parameter and \( \mu = 1/(2 - \nu) \). Then Eq. (1) can be formally linearized:

\[ u'' + \frac{2}{x} u' + \varepsilon^{-\mu} a u = 0, \quad x \in [0, \pi]. \]

To remove the singularity of the kernel of the integral equation obtained below, we denote \( v = u'(x)/x \) and
transform Eq. (3) in the following way:

\[ x v'(x) + 3 v(x) - \varepsilon^{-\mu} a(x) \int_0^x \xi v(\xi) d\xi = \varepsilon^{-\mu} a(x), \quad x \in [0, \pi]. \]

Then we seek a solution in the form

\[ v = \frac{c}{x} \psi(x) + (A \psi)(x), \]

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where \( \psi \) is a new unknown quantity, \( c \) is a constant,

\[
A. = \int_{-\pi}^{\pi} k(x, \xi) d\xi, \quad k = \frac{1}{\pi} \frac{1 - \rho^2}{1 - 2 \rho \cos(x - \xi) + \rho^2}, \quad 0 < \rho < 1. \tag{6}
\]

By substituting (5) into (4), we obtain the equation

\[
c \psi'(x) + \frac{2c}{x} \psi(x) + 3 (A \psi)(x) + x (A' \psi)(x)
- e^{-\mu} a(x) \int_{0}^{x} [c \psi(\xi) - \xi (A \psi)(\xi)] d\xi = e^{-\mu} a(x), \quad x \in [0, \pi]. \tag{7}
\]

Comparing the intervals of definition of \( \psi \) and Eq. (7), we conclude that Eq. (7) may have many solutions. In this connection, we try to satisfy both the mentioned equation and the following one:

\[
\psi(x) = \lambda (A \psi)(x) + f(x), \quad x \in [0, \pi]; \tag{8}
\]

here, \( \lambda \) is a parameter and the function \( f \) is determined by the condition of its equality to the free term of Eq. (7) with \( \psi'(x) \) preliminarily eliminated by using (8).

Equation (7) takes the form

\[
\psi(x) = (B \psi)(x) + f(x), \quad x \in [0, \pi], \tag{9}
\]

where

\[
B. = -3 (A')(x) - (c \lambda + x) (A')(x) + e^{-\mu} a(x) \int_{0}^{\pi} [c + \xi (A')(\xi)] d\xi,
\]

\[
f'(x) + \frac{2}{x} f(x) = \frac{1}{c} e^{-\mu} a(x).
\]

This and relation \( f(0) = 0 \) yield

\[
f = \frac{e^{-\mu}}{cx^2} \int_{0}^{x} \xi^2 a(\xi) d\xi. \tag{11}
\]

2. Analysis of Transformations

Let us examine the existence of representation (5) in the sense of solvability of Eq. (8).

**Lemma 1.** For functions \( \psi(x), x \in [0, \pi] \), representable by Fourier series, there exists a function \( \psi \) on \( x \in [-\pi, 0] \) for which Eq. (8) has a solution in \( L_2[0, \pi] \).

**Proof.** We represent the considered equation in the form