ON STABILITY OF AN $N$TH-ORDER EQUATION IN A CRITICAL CASE

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We obtain sufficient conditions for the Lyapunov stability of the trivial solution of a nonautonomous $n$th-order equation in the case where the root of the boundary characteristic equation is equal to zero and has multiplicity greater than one.

1. Statement of the Problem

We study the Lyapunov stability [1] of the trivial solution of a differential equation of the form

$$y^{(n)} + \sum_{s=1}^{n-1} p_s(t)y^{(n-s)} + p_n(t)y = F(t, y, y', \ldots, y^{(n-1)}),$$

$$t \in \Delta = [a, \omega], \quad -\infty < a < \omega \leq +\infty, \quad p_s : \Delta \to \mathbb{C}, \quad s = 1, n,$$

as $t \uparrow \omega$. Here, $\mathbb{C}$ and $\mathbb{C}^n$ are the set of complex numbers and an $n$-dimensional complex Euclidean space, respectively, and the following conditions are satisfied:

(a) $p_s \in C^2_\Delta, \quad p_s \equiv \pi^s [p^0_s + a_s(1)], \quad t \uparrow \omega, \quad \pi : \Delta \to [0, +\infty[, \quad p^0_s \in \mathbb{C}, \quad s = 1, n$;

(b) the equation

$$\lambda^n + \sum_{s=1}^{n} p^0_s \lambda^{n-s} = 0$$

has $n_0$, $1 \leq n_0 \leq n$, roots $\lambda_s^0$ satisfying the condition $\text{Re} \lambda_s^0 = 0, \ s = 1, n_0$, and $n - n_0$ roots $\lambda_k^0$ satisfying the condition $\text{Re} \lambda_k^0 \leq -\gamma, \ \gamma \in [0, +\infty[, \ k = n_0 + 1, n$; 

(c) $|F| \leq L \left( |y| + \sum_{s=1}^{n-1} |y^{(s)}| \right)^{1+\alpha}$, $L \in C_\Delta, \quad L : \Delta \to [0, +\infty[, \quad \alpha \in [0, +\infty[.$

Below, we use the following notation and definitions:

$$L_\Delta \equiv \left\{ f : \Delta \to \mathbb{C}, \int f |dt < +\infty \right\},$$
\[
\Omega = |\pi''|\pi^{-2} + (\pi')^2\pi^{-3} + \sum_{s=1}^{n} \left( |\pi'|\pi^{-s-1}|p_s| + |p_s|^2 + |\pi'|\pi^{-s} - p_s^0|^2 \right),
\]

\[
\Omega_0 = L\pi^{-n}(\pi + \pi^{-1})^{1+\alpha}, \quad \Omega_s \equiv \Omega_0 \pi^{-n_s-1}, \quad s = \overline{2,s_0},
\]

\[
\Omega_s^r \equiv \pi^{-s_n} |\pi''|\pi^{-2} + |\pi'|\pi^{-2}\pi^{-s_{s+1}} + (\pi')^2\pi^{-3}\pi^{-s_{s+1}}
\]

\[
+ |\pi'\pi_{s+1}^r| (\pi_s\pi_{s+1})^{-2} + \sum_{k=1}^{n} \left( |p'\pi^{-k} + |\pi'\pi^{-k-1}|p_k| + |\pi'_k|\pi^{-k} - p_k^0| \right), \quad s = \overline{1,s_0},
\]

\[
X_{n_s} = \text{col}(x_{1n_s}, \ldots, x_{n_{n_s}}), \quad X = \text{col}(X_{n_1}, \ldots, X_{n_{s_0}}), \quad Z = \text{col}(z_1, \ldots, z_n),
\]

\[
Y_{n_s} = \text{col}(y_{1n_s}, \ldots, y_{n_{n_s}}), \quad Y = \text{col}(Y_{n_1}, \ldots, Y_{n_{s_0}}),
\]

\[
\|X_{n_s}\| = \sum_{k=1}^{n_s} |x_{kn_s}|, \quad \|X\| = \sum_{s=1}^{s_0} \|X_{n_s}\|,
\]

\[
P_{mn} \text{ is a rectangular } m \times n \text{ matrix, } P_{mm} \equiv P_m,
\]

\[
\|P_{mn}\| = \sum_{s=1}^{m} \sum_{k=1}^{n} |p_{sk}|
\]

if \(\|P_{mn}\| = \|p_{sk}\|, s = \overline{1,m}, k = \overline{1,n}\), and \(E_m\) and \(H_m\) are the identity matrix and the \(m \times m\) matrix of shifts, respectively.

**Definition 1.** The differential equation (1) possesses property St as \(t \uparrow \omega\) if, for any \(\varepsilon \in ]0, +\infty[,\) there exist \(\delta_\varepsilon \in ]0, \varepsilon[\) and \(T_\varepsilon \in \Delta\) such that any solution \(y = y(t)\) of (1) with initial condition \(y(T_\varepsilon) \leq \delta_\varepsilon \pi(T_\varepsilon), |y(t)|, |y(s-1)(t)| < \varepsilon \pi^n, \quad s = \overline{2,n}\) satisfies the inequalities \(|y(t)| < \varepsilon \pi^n\) and \(|y(s-1)(t)| < \varepsilon \pi^s, \quad s = \overline{2,n}\), for all \(t \in [T_\varepsilon, \omega[\).

**Definition 2.** The differential equation (1) possesses property As St as \(t \uparrow \omega\) if the conditions of Definition 1 are satisfied and \(\pi^{-1}y(t) = o(1)\) and \(\pi^{-s}y(s-1)(t) = o(1)\) as \(t \uparrow \omega, \quad s = \overline{2,n}\).

**Definition 3.** The differential equation (1) possesses property St as \(t \uparrow \omega\) if the conditions of Definition 1 are not satisfied.

**Definition 1'.** A differential system of the form
\[
X' = S(t, X), \quad S(t, \overline{0}) = \overline{0}, \quad \overline{0} = \text{col}(0, \ldots, 0),
\]
possesses property St as \(t \uparrow \omega\) if, for any \(\varepsilon \in ]0, +\infty[,\) there exist \(\delta_\varepsilon \in ]0, \varepsilon[\) and \(T_\varepsilon \in \Delta\) such that any solution \(X = X(t)\) of (2) with initial condition \(\|X(T_\varepsilon)\| < \delta_\varepsilon\) satisfies the inequality \(\|X(t)\| < \varepsilon\) for all \(t \in [T_\varepsilon, \omega[\).