Novel Perturbation Expansion for the Langevin Equation

Carl Bender, 1 Fred Cooper, 2 L. M. Simmons, Jr., 3 Pinaki Roy, 4 and Greg Kilcup 5

Received December 11, 1990

We discuss the randomly driven system

\[ \frac{dx}{dt} = -W(x) + f(t), \]

where \( f(t) \) is a Gaussian random function or stirring force with \( \langle f(t) f(t') \rangle = 2 \delta(t - t') \), and \( W(x) \) is of the form \( g x^{1+\delta} \). The parameter \( \delta \) is a measure of the nonlinearity of the equation. We show how to obtain the correlation functions \( \langle x(t) x(t') \cdots x(t(n)) \rangle_f \) as a power series in \( \delta \). We obtain three terms in the \( \delta \) expansion and show how to use Padé approximants to analytically continue the answer in the variable \( \delta \). By using scaling relations, we show how to get a uniform approximation to the equal-time correlation functions valid for all \( g \) and \( \delta \).

KEY WORDS: Langevin equation; delta expansion; nonlinear; perturbation expansion; scaling relations.

1. INTRODUCTION

Recently a new perturbative technique, the \( \delta \) expansion, was proposed to solve nonlinear problems in physics.1-3) The technique involves replacing, in a differential equation, nonlinear terms such as \( x^3 \) by \( x^{1+2\delta} \) and expanding this term in powers of \( \delta \):

\[ x^{1+2\delta} = x \sum_{n=0}^{\infty} \delta^n \frac{(\ln x^2)^n}{n!} \]  

(1.1)

We are thus able to obtain a solution to the differential equation as a power series in \( \delta \). The parameter \( \delta \) is a measure of the nonlinearity of the

1 Physics Department, Washington University, St. Louis, Missouri 63101.
2 Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.
3 Santa Fe Institute, Santa Fe, New Mexico 87501.
4 Electronics Unit, India Statistical Institute, Calcutta 700035, India.
5 Department of Physics, Ohio State University, Columbus, Ohio 43210.
theory. When $\delta = 0$ the theory is linear and typically can be solved in closed form. As $\delta$ increases from zero, the nonlinearity turns on smoothly. Typically the $\delta$ series has a finite radius of convergence. Since we are interested in $\delta = 1, 2, \text{etc.}$, we need a way of analytically continuing the series obtained to large $\delta$. To do this, we will employ Padé approximants.\(^{(4)}\)

The first nontrivial Padé approximant, the $[1, 1]$ Padé, requires calculating terms up to order $\delta^2$ in this expansion. We will also utilize a scaling argument to obtain the correct functional dependence of the correlation functions on the coupling constant $g$ for all values of $\delta$.

In this paper we will be studying the one-dimensional Langevin equation

$$\frac{dx}{dt} = W(x) + f(t), \quad W(x) = gx^{1+2\delta}$$

(1.2)

[For this equation to be well defined for arbitrary $\delta$ and negative $x$ we interpret $W$ as follows: $W(x) = gx(x^2)^{\delta}$.]

The stirring force $f(t)$ is a random function described by Gaussian statistics, i.e., it is described by a joint probability functional

$$P[f] = N \exp\left[ -\frac{1}{2} \int_{t_0}^{\infty} dt \, dt' f(t) S(t, t') f(t') \right]$$

(1.3)

with

$$\int P[f] \mathcal{D}f = 1$$

Choosing white noise,

$$S^{-1}(t, t') = 2\delta(t - t')$$

we have that

$$\langle f(t) f(t') \rangle = \int \mathcal{D}f \, P[f] \, f(t) f(t') = S^{-1}(t, t') = 2\delta(t - t')$$

(1.4)

where $\mathcal{D}f$ denotes functional integration.

There are two ways to determine the correlations in $x(t)$ resulting from the statistics of the forcing term. One way is to solve directly for $x(t)$ in terms of $f(t)$ and then use (1.4). The other is to make a change of variables in the functional integral (1.4) to obtain a path integral in the variable $x(t)$. One has\(^{(5,6)}\)

$$\langle x(t) x(t') \rangle_f = \int \mathcal{D}x \, P[x] \, x(t) x(t') = \int \mathcal{D}x(t) \, P[f(x)] \, x(t) x(t') \det \left| \frac{\delta f}{\delta x} \right|$$

(1.5)