ELEMENTARY AND MULTIELEMENTARY REPRESENTATIONS OF VECTROIDS

K. I. Belousov, L. A. Nazarova, A. V. Roiter, and V. V. Sergeichuk

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To the memory of Maurice Auslender

We prove that every finitely represented vectroid is determined, up to an isomorphism, by its completed biordered set. Elementary and multielementary representations of such vectroids (which play a central role for biinvolutive posets) are described.

Introduction

Denote by $k$ a fixed algebraically closed field and by $\text{mod } k$ the category of finite-dimensional right vector spaces over $k$. The symbol of a linear map is written to the right.

A vectroid $\mathcal{V}$ (over the field $k$) is a small (the class of objects is a set) subcategory of the category $\text{mod } k$ which is a spectroid in the sense of [1], i.e., satisfies the following conditions:

(i) For each pair of objects $X, Y \in \mathcal{V}$, the set $\mathcal{V}(X, Y)$ of morphisms is a linear subspace in $\text{mod } k(X, Y)$;

(ii) For each $X \in \mathcal{V}$, the ring $\mathcal{V}(X, X)$ contains exactly two idempotents ($0_X \neq 1_X$);

(iii) $\mathcal{V}$ does not contain isomorphic objects.

The value $\sup \{\dim X, X \in \mathcal{V}\}$ is called the dimension $\dim \mathcal{V}$ of $\mathcal{V}$.

Each vectroid $\mathcal{V}$ defines a category (an aggregate in the sense of [1]) $\oplus \mathcal{V} \subseteq \text{mod } k$ whose objects are all finite direct sums $X_1 \oplus \ldots \oplus X_m (X_i \in \mathcal{V}, m \geq 0)$. The category $\oplus \mathcal{V}$ (as well as any subcategory in $\text{mod } k$) can be regarded as a faithful module over itself [1].

A triple $(U, f, X)$ consisting of the spaces $U \in \text{mod } k$ and $X \in \oplus \mathcal{V}$ and a linear map $f: U \to X$ is called a representation of $\mathcal{V}$ ([1], 4.1, [2]). The morphism $(U, f, X) \to (U', f', X')$ is a pair $(\varphi, \xi)$ that consists of a linear map $\varphi: U \to U'$ and a morphism $\xi: X \to X'$ of the category $\oplus \mathcal{V}$ such that $\varphi f' = f' \xi$. Representations form the aggregate denoted by $\text{Rep } \mathcal{V}$. A vectroid is called finitely represented if $\text{Rep } \mathcal{V}$ has finitely many indecomposable nonisomorphic objects.

Within the notation of ([1], 4.1), the category $\text{Rep } \mathcal{V}$ coincides with the category $(\oplus \mathcal{V})^k$. In some cases, we shall consider the category $M^k$ of representations of an arbitrary module $M$ (not necessarily faithful) over an aggregate (see Appendix at the end of Introduction).

If $\dim \mathcal{V} = 1$, then $\mathcal{V}$ is completely determined by the following partial ordering of the set $\text{Ob } \mathcal{V}$: $X \leq Y$ if $\mathcal{V}(X, Y) \neq 0$. The category $\text{Rep } \mathcal{V}$ can be naturally identified with the category of representations of this poset ([1], 4.1, [5]). The criterion of finite representability of posets was obtained in [6].

On the other hand, it was proved in ([1], 4.2, 4.3) and ([3], 9.1, 9.4) that the category $\text{mod } \Lambda$ of representations of an arbitrary finite-dimensional algebra $\Lambda$ over $k$ coincides with the category of representations of a certain vectroid $\mathcal{V}'$ in the following sense: There exists an injective indecomposable $\Lambda$-module $P$ such that the category of all $\Lambda$-modules that do not contain $P$ as a direct summand is equivalent to $\text{Rep } \mathcal{V}'$. 


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Thus, the theory of representations of vectroids can be regarded as a generalization of the theory of representations of posets and the theory of representations of finite-dimensional algebras.

We define the radical of a vectroid $\mathcal{V}$ as the ideal of $\mathcal{V}$ generated by the spaces $\text{Rad}_{\mathcal{V}}(X, Y)$ of uninvertible morphisms from $\mathcal{V}(X, Y)$ for all $X, Y \in \mathcal{V}$. It is obvious that $\mathcal{V}(X, Y) = \text{Rad}_{\mathcal{V}}(X, Y)$ for $X \neq Y$ and $\mathcal{V}(X, X) = k 1_X \oplus \text{Rad}_{\mathcal{V}}(X, X)$.

The set $\{(n_i^X, (f_i^{XY})\}$ that consists of the bases $(n_1^X, n_2^X, \ldots, n_{\dim(X)}^X)$ of spaces $X \in \mathcal{V}$ and the bases $(f_1^{XY}, f_2^{XY}, \ldots)$ of spaces $\text{Rad}_{\mathcal{V}}(X, Y)$ is called the basis of the vectroid $\mathcal{V}$. The maximal rank of the linear maps $f_i^{XY}$ is called the rank of the basis. The basis $(n_1^X, n_2^X, \ldots, n_{\dim(X)}^X)$ of an object $X \in \mathcal{V}$ is called triangular if the family $\{(n_i^X), i = 1, \dim X \mid (n_i^X) \neq 0\}$ is linearly independent for any $j \in \mathbb{N}$, where the bar means the transition to the factor space $X/X_{\text{Rad}}^i(\mathcal{V}, X, X)$. The basis $\{(n_i^X), (f_i^{XY})\}$ of $\mathcal{V}$ is called triangular if each basis $(n_i^X), X \in \mathcal{V}$, is triangular. A basis is scalarly multiplicative if the element $n_i^X f_i^{XY}$ is equal to $\lambda n_p^Y$, $\lambda \in k$, for all $n_i^X, f_i^{XY}$, and it follows from the relations $n_i^X f_i^{XY} = \lambda n_p^Y$ and $n_i^X f_i^{XY} = \mu n_p^Y$, $\lambda, \mu \in k^*$, that $i = j$. A scalarly multiplicative basis is called multiplicative if each element $n_i^X f_i^{XY}$ is equal to either 0 or $n_p^Y$ ([1], 4.10). Every finitely represented vectroid has a multiplicative basis whose rank does not exceed two [4].

A vectroid $\mathcal{V}$ is called a chain vectroid if, for every $X \in \mathcal{V}$, submodules of the module $X_{\mathcal{V}(XX)}$ are linearly ordered with respect to the inclusions

$$X = X_1 \supset X_2 \supset \ldots \supset X_{\dim X} \supset 0.$$ 

In this case, all these submodules are cyclic, $X_i = m_i^X \mathcal{V}(X, X)$, and $m_1^X, m_2^X, \ldots, m_{\dim X}^X$ is a triangular basis of $X \in \mathcal{V}$ (see Lemma 1).

It is known that if $\mathcal{V}$ is a finitely represented vectroid, then $\mathcal{V}$ is a chain vectroid and $\dim \mathcal{V} \leq 3$ ([1], 4.7 and 4.8).

For an arbitrary chain vectroid $\mathcal{V}$, we construct the poset

$$S(\mathcal{V}) = \bigcup_{X \in \mathcal{V}} \{X_1, X_2, \ldots, X_{\dim X}\},$$

setting $X_i \leq Y_j$ if $m_i^X \varphi = m_j^Y$ for some $\varphi \in \mathcal{V}(X, Y)$.

The number $\text{def}(\mathcal{V}) = \sup \{\text{def}(X, Y) \mid X, Y \in \text{Ob} \mathcal{V}\}$, where

$$\text{def}(X, Y) = |\{(X_i, Y_j) \mid X_i < Y_j\}| - \dim \text{Rad}(X, Y),$$

is called the defect of $\mathcal{V}$. According to [4], we have $\text{def} \mathcal{V} \leq 1$ for all finitely represented vectroids $\mathcal{V}$ (see Sec. 2).

If $\text{def} \mathcal{V} = 0$, then $\mathcal{V}$ has a multiplicative basis of rank one and the category $\text{Rep} \mathcal{V}$ coincides with the category of representations of a weakly completed poset $S(\mathcal{V})$ (see Sec. 1). The criterion of finite representability of weakly completed posets and the classification of their indecomposable representations (in the case of finite representability) are given in [9] (see also [10]).

Let $\text{def} \mathcal{V} = 1$ and $\dim \mathcal{V} \leq 2$. In this case, representations of vectroids $\mathcal{V}$ are identified with representations of a certain poset $S(\mathcal{V})$ with additional structure (the structure of a biinvolutive poset). In this case, $\mathcal{V}$ is finitely represented if and only if a certain poset $S(S(\mathcal{V}))$ constructed for a biinvolutive poset $S([1], 5.8)$ is finitely represented. This criterion was formulated in [1] and proved in [7, 8].