INTEGRATION OF A SINGULARLY PERTURBED DEGENERATE LINEAR SYSTEM

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We present a method for finding a solution of a linear homogeneous system with degenerate matrix and a small parameter with the derivative.

1. Consider the system

\[ \varepsilon^h B(t) \dot{x} = A(t, \varepsilon)x, \]

where \( A(t, \varepsilon) = \sum_{s \geq 0} \varepsilon^s A_s(t) \), \( B(t) \) are \((n \times n)\) matrices, \( t \in [t_0, T] \), \( 0 < \varepsilon \leq \varepsilon_0 < 1 \), \( \det B(t) \equiv 0 \), and the pencil of matrices \( A_0(t) - \lambda B(t) \) is regular.

Note that if \( \det B(t) \neq 0 \) for \( t \in [t_0, T] \), then system (1) has a general solution of the form

\[ x(t, \varepsilon) = R(t, \varepsilon) \exp \left( \int_{t_0}^{t} \Lambda(\tau, \varepsilon) d\tau \right) c, \]

where \( R(t, \varepsilon) \) and \( \Lambda(t, \varepsilon) \) are \((n \times n)\) matrices and \( c \) is a constant \( n \)-dimensional vector. In this case, if the characteristic equation

\[ \det (A_0(t) - \lambda B(t)) = 0 \]

has simple roots, then

\[ R(t, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s R_s(t), \]

\[ \Lambda(t, \varepsilon) = \sum_{s=0}^{h-1} \varepsilon^s \Lambda_s(t) = \text{diag} \{ \lambda_1(t, \varepsilon), \ldots, \lambda_n(t, \varepsilon) \}. \]

If Eq. (3) has multiple roots associated with multiple elementary divisors, then, in (4), we have an expansion in fractional powers of \( \varepsilon \).

System (1) was first considered for \( \det B(t) \equiv 0 \) in [1]; it was shown that if Eq. (3) has \( r_1 \) simple roots \( \lambda_i(t) \), \( i = 1, r_1 \), \( 1 \leq r_1 < n \), then system (1) has \( r_1 \) particular solutions of the form

\[ x_k(t, \varepsilon) = \varphi_k(t, \varepsilon) \exp \left( \int_{t_0}^{t} \omega_k(\tau, \varepsilon) d\tau \right), \quad k = 1, r_1, \]
where

$$\varphi_k(t, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \varphi_k^{(s)}(t), \quad \omega_k(t, \varepsilon) = \lambda_k(t) + \sum_{s=1}^{h-1} \varepsilon^s \omega_k^{(s)}(t).$$

If Eq. (3) has an $r_1$-multiple root $\lambda_0(t)$ associated with multiple elementary divisors, then system (1) also has $r_1$ particular solutions of the form (5) with expansions in fractional powers of $\varepsilon$. The problem of existence of other solutions of system (1) remained open for a long time. This problem was completely solved in [2]. In the present paper, we show that, under certain conditions, in addition to solutions of the form (5) with

$$\omega_k(t, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \omega_k^{(s)}(t)$$

(which are called solutions of the first type), there exist solutions of the second type, namely,

$$x(t, \varepsilon) = \left( \sum_{s=0}^{\infty} \mu^s v^{(s)}(t) \right) \exp \left( \varepsilon^{-h} \int_{t_0}^{t} \eta^{-1}(\tau, \mu) d\tau \right),$$

(6)

where

$$\eta(t, \varepsilon) = \sum_{s=0}^{\infty} \mu^s \eta^{(s)}(t), \quad \mu^m = \varepsilon,$$

and $m$ is a certain integer. As shown in [2], solutions of this or another form are determined by elementary divisors of the matrix pencil $A_0(t) - \lambda B(t)$. Namely, solutions of the form (5) correspond to “finite” elementary divisors, while solutions of the form (6) correspond to “infinite” elementary divisors.

It is shown in [3] that if the roots of Eq. (3) and all elementary divisors of the pencil $A_0(t) - \lambda B(t)$ preserve constant multiplicity on $[t_0, T]$, then there exists a pair of nonsingular matrices $P(t)$ and $Q(t)$ that reduces the pencil to the canonical form

$$P(t)(A_0(t) - \lambda B(t))Q(t) = \Omega(t) - \lambda H,$$

(7)

$$\Omega(t) = \begin{pmatrix} W(t) & 0 \\ 0 & E_2 \end{pmatrix}, \quad H = \begin{pmatrix} E_1 & 0 \\ 0 & I \end{pmatrix},$$

(8)

where $W(t) = \text{diag} \{ \lambda_1(t), \ldots, \lambda_{r_1}(t) \}$ in the case of simple roots of Eq. (3) or it is a Jordan $(r_1 \times r_1)$ matrix, $E_1$ and $E_2$ are identity matrices of dimensions $r_1$ and $r_2 = n - r_1$, respectively, and $I$ is a nilpotent Jordan matrix determined by “infinite” elementary divisors of the pencil $A_0(t) - \lambda B(t)$.

Taking (8) into account, one can conclude that solutions (5) are determined by the pair $(W(t), E_1)$, and solutions (6) are determined by $(E_2, I)$.

Furthermore, it is shown in [2] that if rank $B(t) = r_1$, i.e., the “rank--degree” condition [4] is satisfied, then system (1) has only solutions of the form (5). Therefore, we can represent the general solution in the form (2), where $R(t, \varepsilon)$ and $\Lambda(t, \varepsilon)$ are $(n \times r_1)$ and $(r_1 \times r_1)$ matrices, respectively, and $c$ is an $r_1$-dimensional vector. If rank $B(t) = r > r_1$, then solutions of the form (6) appear. If Eq. (3) has no roots, i.e.,

$$\det (A_0(t) - \lambda B(t)) = \det A_0(t) \neq 0,$$

(9)