BRIEF COMMUNICATIONS

ON MAJORANTS IN THE HARDY–LITTLEWOOD THEOREM
FOR HIGHER DERIVATIVES

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We establish conditions for majorants under which the classical Hardy–Littlewood theorem for the class of functions analytic in a disk is true in terms of derivatives of arbitrary fixed order.

Denote by $D$ a unit disk in the complex plane $C$. As in [1], we say that an increasing function $\lambda(t) : [0, \infty) \to [0, \infty)$, $\lambda(0) = 0$, is a majorant if $\lambda'(t)$ decreases. Other definitions of majorant can be found in [2–5].

Let $f : D \to C$ and let $\lambda(t)$ be a majorant. We write $f \in \text{Lip}_\lambda(D)$ if one can find a constant $M < \infty$ such that $|f(z_1) - f(z_2)| \leq M\lambda(|z_1 - z_2|)$ for all $z_1, z_2 \in D$. By $\|f\|_\lambda$ we denote the infimum of all such $M$.

In the notation introduced above, the classical Hardy–Littlewood theorem can be formulated as follows: If $f$ is analytic in $D$, $\lambda(t) = t^\alpha$, $0 < \alpha \leq 1$, and $f \in \text{Lip}_\lambda(D)$, then there exists a constant $A > 0$ such that

$$|f'(z)| \leq A\lambda'(1-|z|)$$

(1)

for all $z \in D$; the constant $A$ depends only on $\alpha$ and $\|f\|_\lambda$ [6, 7]. In [1], it was proved that relation (1) is equivalent to the condition

$$\limsup_{t \to 0^+} \frac{\lambda(t)}{t\lambda'(t)} < \infty.$$

The following problem arises: Under what conditions imposed on the majorant does the corresponding result [1] remain true for higher derivatives of $f$? We solve this problem by introducing regularly monotone majorants. A real function is called regularly monotone on a certain interval if this function and all its derivatives preserve their signs on this interval [8]. A majorant is called regularly monotone in the interval $(0, \infty)$ if $\lambda^{(k)}(t) \neq 0$ and $|\lambda^{(k)}(t)|$ decreases for $t > 0$ and $k = 1, 2, \ldots$.

Theorem 1. For $k = 1, 2, \ldots$, the following statements are equivalent:

(i) If $f$ is analytic in $D$, $\lambda(t)$ is a regularly monotone majorant of this function on the interval $(0, \infty)$, and $f \in \text{Lip}_\lambda(D)$, then there exists a constant $A > 0$ depending only on $\lambda$ and $\|f\|_\lambda$ such that the following estimate holds for all $z \in D$:

$$|f^{(k)}(z)| \leq A|\lambda^{(k)}(1-|z|)|$$

(2)
(ii) for a regularly monotone majorant \( \lambda(t) \) of a function \( f \) analytic in \( D \), the following relation is true:
\[
\limsup_{t \to 0^+} \frac{\lambda(t)}{t^k |\lambda^{(k)}(t)|} < \infty.
\] (3)

Remark. In the case \( k = 1 \), we arrive at Theorem 1.3 in [1].

Proof of Theorem 1. For simplicity of analytic calculation, we prove Theorem 1 for \( k = 2 \). In the general case, the proof is analogous.

First, we prove that statement (ii) implies statement (i). If relation (3) holds, then there exist positive constants \( t_0 \) and \( C_0 \) such that
\[
\lambda(t)t^{-2} \leq C_0 |\lambda''(t)|, \quad t \in (0, t_0).
\] (4)

Inequality (4) also holds for all \( t \in (0, 1] \) if the constant \( C_0 \) is replaced by the constant \( C_1 = \max \{ C_0, \lambda(t_0)t_0^{-2}|\lambda''(t)|^{-1} \} \). We fix \( z \in D \) and set \( 0 < R < 1 - |z| \). Since \( f \in \text{Lip}_\lambda(D) \), by using the integral Cauchy formula and condition (4), we get
\[
|f''(z)| = \left| \frac{1}{\pi i} \int_0^{2\pi} \frac{f(z + R e^{i\theta}) - f(z)}{R^3 e^{3i\theta}} R e^{i\theta} i d\theta \right| \leq 2C_1 \|f\|_{\lambda, |\lambda''(R)|}.
\]

Passing to the limit as \( R \to 1 - |z| \) in the last inequality, we establish (2) for \( k = 2 \).

We prove that statement (i) implies statement (ii) by contradiction. For this purpose, we assume that (3) is not valid and show that there exists a function \( f \in \text{Lip}_\lambda(D) \) for which statement (i) is not true. Since, by assumption,
\[
\limsup_{t \to 0^+} \frac{\lambda(t)}{t^2 |\lambda''(t)|} = +\infty,
\]
there exists a monotonically decreasing sequence \( \{t_j\}_{j=1}^\infty \) of values of the argument \( t \in (0, 1] \) for which
\[
\frac{\lambda(t_j)}{t_j^2 |\lambda''(t_j)|} \geq 2^{3j}, \quad j = 1, 2, \ldots.
\] (5)

We define a function \( f(z) \) analytic in \( D \) by setting
\[
f(z) = \sum_{j=1}^{\infty} a_j z^{n_j},
\] (6)
where \( a_j = 2^{-j} \lambda(t) \) and \( n_j \) is the integer part of a positive number \( t_j^{-1} \). One can easily verify that this series has the unit circle of convergence. Let us show that the function \( f \) thus constructed belongs to \( \text{Lip}_\lambda(D) \). The following estimate holds for \( l = 0, 1, 2, \ldots \) and arbitrary \( z_1, z_2 \in D \):
\[
|z_1^l - z_2^l| \leq \min \{ 2, l |z_1 - z_2| \}.
\] (7)

By setting \( t = |z_1 - z_2| \) and using (7), we obtain the estimate