The Relative Consistency of System RRC* and Some of its Extensions*

Abstract. We present a relative consistency proof for second order system RRC* and for certain important extensions of this system. The proof proceeds as follows: we prove first the equiconsistency of the strongest of such extensions (viz., system HRRC* + \((3/CP**)) with second order system T\(\Lambda\). Now, N. Cocchiarella has shown that T\(\Lambda\) is relatively consistent to system T\(\lambda\) + Ext; clearly, it follows that HRRC* + \((3/CP**)) is relatively consistent to T\(\lambda\) + Ext. As an immediate consequence, the relative consistency of RRC* and the other extensions also follows, being all of them subsystems of HRRC* + \((3/CP**))

1. Introduction

The relative consistency of the second order logical systems RRC*, HRRC*, TRRC*, and HRRC* + \((3/CP**)) is the main topic of this paper. First, we prove the equiconsistency of HRRC* + \((3/CP**)) with Cocchiarella's system T\(\lambda\). Since the latter system is relatively consistent to T\(\lambda\) + Ext (as shown in [2]), the relative consistency of the aforementioned second order logical systems to T\(\lambda\) + Ext follows, because they are subsystems of HRRC* + \((3/CP**))\(\uparrow\).

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1Systems RRC* and HRRC* were formulated in [3] and systems TRRC*, HRRC* + \((3/CP**)) in [5]. For technical details concerning T\(\lambda\) and T\(\lambda\) + Ext cf, [2]. System RRC* was claimed by Nino Cocchiarella, in [3], to be absolutely consistent. However, as the author in private conversation pointed out to Cocchiarella (and acknowledged by Cocchiarella himself in [4]), his proof in [3] is not correct. In his proof, he describes an operation which is supposed to map wffs of RRC* into wffs of \(\lambda T\). However, such an operation cannot map, for instance, "t = t" into a wff of \(\lambda T\) when "t" is a lambda abstract term containing the identity sign. Unlike T\(\lambda\), the identity sign is not permitted to occur in the \(\lambda\)-abstracts of \(\lambda T\), just as it not permitted to occur in the comprehension principle of T\(\lambda\).

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2. Syntax

We begin by describing the syntax. We take a language $L$ to be a countable set of individual and predicate constants. We assume the availability of denumerable many individual variables as well as many $n$-place predicate variables (for each natural number $n$). We shall also use $x, y, z$ and $w$, with or without numerical subscripts, to refer in the metalanguage to individual variables and $F^n, G^n$ and $R^n$ to refer to $n$-place predicate variables. We shall usually drop the superscript when the context makes clear the degree of a predicate variable or when otherwise does not matter what degree it is. For convenience, we shall also use $u$ in order to refer to variables in general.

As primitive logical constants we take $\rightarrow, =, \neg, \lambda, \forall$ and $\forall^j$ (for each natural number $j > 0$).

Given a language $L$, we define recursively the set of meaningful expressions of type $n$ of $L$, (in symbols, $ME_n(L)$) as follows:

1. every individual variable or constant is in $ME_0(L)$
   every $n$-place predicate variable or constant is in both $ME_{n+1}(L)$ and $ME_0(L)$

2. if $a, b \in ME_0(L)$, then $(a = b) \in ME_1(L)$

3. if $\pi \in ME_{n+1}(L)$ and $a_1, \ldots, a_n \in ME_0(L)$, then $\pi(a_1, \ldots, a_n) \in ME_1(L)$

4. if $\delta \in ME_1(L)$ and $x_1, \ldots, x_n$ are pairwise distinct individual variables, then $[\lambda x_1, \ldots, x_n \delta] \in ME_{n+1}(L)$

5. if $\delta \in ME_1(L)$, then $\neg \delta \in ME_1(L)$

6. if $\delta, \sigma \in ME_1(L)$, then $(\delta \rightarrow \sigma) \in ME_1(L)$

7. if $\delta \in ME_1(L)$, $x$ is an individual variable, $F$ is a predicate variable, and $j$ is a positive integer, then $(\forall x)\delta, (\forall^j x)\delta, (\forall F)\delta$ and $(\forall^j F)\delta$ in $ME_1(L)$

8. if $\delta \in ME_1(L)$, then $[\lambda \delta] \in ME_0(L)$

9. if $n > 1$, then $ME_n(L) \subseteq ME_0(L)$

We set $ME(L) = \bigcup_{n \in \omega} ME_n(L)$, that is, the set of meaningful expressions of $L^2$. We shall use $\delta, \mu, \sigma, \pi$ and $\alpha$ to refer to meaningful expressions of $L$.

\footnote{Note that predicates expressions are allowed to occur in subject positions of other}