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A Note on the
ω-Completeness
Formalization

Abstract. The paper studies two formal schemes related to ω-completeness.

Let $S$ be a suitable formal theory containing primitive recursive arithmetic and let $T$ be a formal extension of $S$. Denoted by (a), (b) and (c), respectively, are the following three propositions (where $\alpha(x)$ is a formula with the only free variable $x$): (a) (for any $n$) $(\vdash_T \alpha(n))$, (b) $(\vdash_T \forall x \Pr_T(\neg \alpha(x))$ and (c) $(\vdash_T \forall x \alpha(x)$ (the notational conventions are those of Smoryński [3]). The aim of this paper is to examine the meaning of the schemes which result from the formalizations, over the base theory $S$, of the implications (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (b), where $\alpha$ ranges over all formulae. The analysis yields two results over $S$: 1. the schema corresponding to (b) $\Rightarrow$ (c) is equivalent to $\neg \text{Cons}_T$ and 2. the schema corresponding to (a) $\Rightarrow$ (b) is not consistent with $1\text{-CON}_T$. The former result follows from a simple adaptation of the ω-incompleteness proof; the second is new and is based on a particular application of the diagonalization lemma.

Let $S$ be a suitable formal theory containing primitive recursive arithmetic (for instance PRA itself) and let $T$ be a formal extension of $S$. Presupposed are, moreover, the notational conventions of Smoryński [3]. Denoted by (a), (b) and (c), respectively, are the following three propositions (where $\alpha(x)$ is a formula with the only free variable $x$): (a) (for any $n$) $(\vdash_T \alpha(n))$, (b) $(\vdash_T \forall x \Pr_T(\neg \alpha(x))$ and (c) $(\vdash_T \forall x \alpha(x)$. Consider, now, the implications (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (c) e (a) $\Rightarrow$ (b), where $\alpha$ ranges over all formulae. These are expressions which belong to the metalevel of $T$ and have particular meaning. For example, it is evident that the implication (a) $\Rightarrow$ (c) coincides with the ω-completeness of $T$. The aim of this paper, however, is to investigate the meaning of the two other implications; meaning which emerges only when one passes to the formal level of the object-language of the theory $S$. Thus the article examines the schemes that result from the formalizations, over the base theory $S$, of the implications (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (b), where $\alpha$

1In particular, I have borrowed the following from Smoryński: the definitions of the predicates $\Pr_T$ and $\Pr_T$; the substitution function (restricted to numerals) within the provability predicate $\Pr_T(\neg \alpha(x))$; the symbols of the derivability conditions $D1$, $D2$ and $D3$ of Gödel's theorems.

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ranges over all the formulae. The implication \( (a) \Rightarrow (c) \) is also obviously important but, as said, \( (a) \Rightarrow (c) \) coincides with the \( \omega \)-completeness of \( T \); hence the formal schema corresponding to \( (a) \Rightarrow (c) \) is equivalent, over \( S \), to the non-consistency of \( T \).

However, to my knowledge, the other schemes have not been subjected before to specific analysis; consequently, they are thematically studied in this article. The first section examines the schema corresponding to \( (b) \Rightarrow (c) \) and finds that it is equivalent, over \( S \), to \( \neg \text{Cons}_T \). The second section examines the schema corresponding to \( (a) \Rightarrow (b) \) and reaches the conclusion that it is not consistent, over \( S \), with \( 1 - \text{CON}_T \). The former result follows from a simple adaptation of the \( \omega \)-incompleteness proof mentioned above; the latter is instead new and is based on a particular application of the diagonalization lemma.

Once established formally, these results can also be understood in their intuitive form. For this purpose it is sufficient to return to the informal metalinguistic level relative to \( T \), so as to grasp their meaning in terms of the falsity of the implications \( (b) \Rightarrow (c) \) and \( (a) \Rightarrow (b) \). It is assumed, in fact, that \( \neg \) with respect to the standard model of arithmetic \( 1 - \text{CON}_T \) is true. Therefore, not only is the implication \( (a) \Rightarrow (c) \) false – and hence one or the other of the implications \( (b) \Rightarrow (c) \) and \( (a) \Rightarrow (b) \) is false –, but both \( (b) \Rightarrow (c) \) and \( (a) \Rightarrow (c) \) are false as well. It is for this reason that in what follows the two principal results of the present analysis are also paraphrased in terms of the falsity of the respective implications.

1. The relation between \( (b) \) and \( (c) \)

Let the symbol \( (1) \) be the abbreviation for the formal expression (in the language of \( S \)) of \( (b) \Rightarrow (c) \), i. e. \( \Pr_T(\neg\forall x\Pr_T(\neg\alpha(x)^-)\neg) \Rightarrow \Pr_T(\neg\forall x\alpha(x)^-) \). Thus the relation between \( (b) \) and \( (c) \) is characterized by the following theorem.

**Theorem 1.** Over \( S \) \( (1) \) is equivalent to \( \neg \text{Cons}_T \)

The proof results from the first and the third of the following propositions, which correspond to the two directions of the theorem. The second proposition operates only as a lemma for the third. Henceforth \( \gamma \) is the undecidable Gödel's formula.

\(^2\)Cf. Smoryński [3], p. 855.

\(^3\)Note that the occurrence of \( \gamma \) in Proposition 2 is not essential: it could be replaced by the symbol of contradiction \( \perp \). In Smoryński’s proof, in fact, Proposition 2 is formulated in this way and is obtained as an exemplification of Feferman’s lemma cited at p. 847. For alternative demonstrations of the same proposition cf. Girard [2], p. 80 and p. 228.