ASYMPTOTIC EXPANSIONS ASSOCIATED WITH THE JACKKNIFE FUNCTIONAL. II

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This paper is a sequel to part I [Ukr. Mat. Zh., 47, No. 4, 443-452 (1995)]. By using the results of the first part, we obtain the initial terms of the asymptotic expansions of the bias and variance for the jackknife estimator of the variance of the error of observations in a nonlinear regressive model.

6. Theorem on Asymptotic Expansion of Moments

The present paper is a sequel to [11]. Therefore, we continue the numbering of formulas and references and preserve all notation and assumptions. To the conditions of regularity imposed in [11], we add the following condition:

\[ \sup_{\theta \in Q} \mathbb{P}\{n^{1/2} |\hat{\theta}_n - \theta| > H\} \leq c_{14} H^{-m} \]

for \( \hat{\theta} = \hat{\theta}_n, \hat{\theta}_{(-t)}, 1 \leq t \leq n \), with the same constant \( c_{14} = c_{14}(m) \leq \infty \).

The conditions guaranteeing the validity of property VII\((m)\) for the estimator \( \hat{\theta}_n \) can be found in [5-7].

**Theorem 2.** If the model (1) from [11] with \( k = 4 \) and \( m \geq 10 \) satisfies the regularity conditions I\((k + 2)\), II\((k)\), III\((k + 2, m)\), IV, V\((k, m)\), VI\((k)\), and the condition VII\((m)\), then

\[ \sup_{\theta \in Q} |E^\theta_n n^{1/2} (J_n - \sigma^2)| = \begin{cases} o(n^{-1/2}), & m = 10, 11; \\ o(n^{-1}), & m \geq 12, \end{cases} \]

\[ \sup_{\theta \in Q} |D^\theta_n (n^{1/2} (J_n - \sigma^2)) - (\mu_4 - \sigma^4 + 2q\sigma^4 n^{-1})| = o(n^{-1}), \quad m \geq 18. \]

7. Proof of Theorem 2

By using the asymptotic expansion (4) [11] and the methods suggested in [5, 12], one can obtain asymptotic expansions of any moments of the random variable \( n^{1/2} (J_n - \sigma^2) \). Let us study a more special (but interesting for applications) problem of determining the first terms of asymptotic expansions of the first two moments, namely, \( \Delta(J_n) = E^\theta_n n^{1/2} (J_n - \sigma^2) \) and \( S(J_n) = E^\theta_n (J_n - \sigma^2)^2 \). In this case, as a starting point, we take expansion (4) [11] with \( k = 4 \), i.e.,

\[ n^{1/2} (J_n - \sigma^2) = \sum_{v=0}^2 n^{-v/2} G_{vn}(\theta) + n^{-3/2} \hat{R}_{3,n}(\theta), \]

(27)
Consider an event $\Omega_{n}(\theta) = \{ |\hat{R}_{3,n}(\theta)| > c_{16} \log^{5/2} n \}$. Let $\chi(A)$ be the indicator of the event $A$ and let $\bar{\chi} = 1 - \chi$. Uniformly in $\theta \in Q$,

$$
E_{\theta} n^{1/2} (J_{n} - \sigma^{2}) \chi \{ \Omega_{n}(\theta) \} = \sum_{v=0}^{2} E_{\theta} n^{-v/2} G_{v,n}(\theta) \chi \{ \Omega_{n}(\theta) \} + O(n^{-3/2} \log^{5/2} n).
$$

Let us now estimate $E_{\theta} n^{-v/2} G_{v,n}(\theta) \chi \{ \Omega_{n}(\theta) \}$, $v = 0, 1, 2$. In the proof of Theorem 1, we have indicated that the polynomials $G_{v,n}(\theta)$ consist of linear combinations of monomials which contain products of the random variables $b(\alpha_{v})(\theta)$, $\gamma = 1, \ldots, v$, and $\rho_{B} B_{n}^{(\alpha_{1}) \ldots (\alpha_{v}(\rho_{n} + \rho_{A}))}$ with the properties

$$
\sup_{\theta \in Q} P_{\theta}^{n} \left\{ \left| b_{n}(\alpha_{\gamma})(\theta) \right| > c_{17} \log^{1/2} n \right\} = O(n^{-(m-2)/2} \log^{m/2} n),
$$

$$
\sup_{\theta \in Q} P_{\theta}^{n} \left\{ \left| \rho_{B} B_{n}^{(\alpha_{1}) \ldots (\alpha_{\gamma}(\rho_{n} + \rho_{A}))}(\theta) \right| > c_{17} \log^{1/2} n \right\} = O(n^{-(m-2)/2} \log^{m/2} n)
$$

for all $\rho_{B}$ and $\alpha_{\gamma}$ encountered in $G_{v,n}$, $v = 0, 1, 2$. Moreover, the coefficients in these products are uniformly bounded in $n$ and $\theta \in Q$. Let us now introduce a general notation $B_{n}^{(\alpha)}$ for all $\rho_{B} B_{n}^{(\alpha_{1}) \ldots (\alpha_{v}(\rho_{n} + \rho_{A}))}$ and $b(\alpha_{v})$, i.e., $B_{n}^{(\alpha)}$ denotes a random variable such that

$$
\sup_{\theta \in Q} P_{\theta}^{n} \left\{ \left| B_{n}^{(\alpha)}(\theta) \right| > c_{17} \log^{1/2} n \right\} = O(n^{-(m-2)/2} \log^{m/2} n).
$$

Here, $(\alpha)$ is either $(\alpha_{v})$ or $(\alpha_{1}) \ldots (\alpha_{2(\rho_{n} + \rho_{A})})$. Since the number of monomials in $G_{v,n}(\theta)$ does not depend on $n$, to estimate $E_{\theta} n^{-v/2} G_{v,n}(\theta) \chi \{ \Omega_{n}(\theta) \}$, it suffices to consider the analogous mathematical expectation for a single monomial in the polynomial $G_{v,n}(\theta)$. Since

$$
\prod_{\delta=1}^{r+1} B_{n}^{(\alpha_{\delta})}(\theta) \leq \left( \sum_{\delta=1}^{r} \left| B_{n}^{(\alpha_{\delta})}(\theta) \right|^{2} \right)^{r/2}, \quad 2 \leq r \leq v + 1,
$$

it suffices to estimate variables

$$
E_{\theta} \left| B_{n}^{(\alpha)}(\theta) \right|^{r} \chi \{ \Omega_{n}(\theta) \}, \quad |\alpha| = 1, \ldots, v, \quad 2 \leq r \leq v + 1.
$$

We now consider the method of estimating these variables in detail because other estimates required in the proof of Theorem 2 can be obtained similarly. We fix $(\alpha)$ and, for some $\tau > 1$ and $\delta > 0$, we set

$$
\gamma_{jn} = \tau^{j} ( (m - 1 + \delta) \log n )^{1/2}, \quad j = 0, 1, \ldots
$$

Consider a collection of events