We obtain the conditions of essential self-adjointness of Dirichlet operators of Gibbs measures with essential domains consisting of smooth cylindrical functions. It is proved that certain spaces of smooth functions are invariant under the action of the semigroup of the Dirichlet operator.

In recent years, numerous works have appeared dealing with the investigation of quantum lattice systems, in which the spin space of every particle is defined as a compact Riemann manifold [1–3]. For these models, the corresponding stochastic (Glauber) dynamic is given by a semigroup generated by the so-called Dirichlet operator of the Gibbs measure on the infinite product of manifolds. In this case, one of the most important problems is to establish the conditions of the uniqueness of dynamics that corresponds to the essential self-adjointness of the generator of a semigroup.

In the present paper, we obtain the mentioned property of the generator as a result of preservation of certain classes of smooth functions by a semigroup. A similar theorem was announced in [1, 2] without proof and was used to study the properties of corresponding operators.

Consider the following problem. Let $\mathbb{Z}^d$ be a $d$-dimensional lattice, every node $k \in \mathbb{Z}^d$ of which is associated with a compact Riemann manifold $M_k \equiv M$. Consider a family of probability measures $\{\mu_\Lambda\}$ given on the Borel $\sigma$-algebras $\mathcal{B}(M^\Lambda)$, where $M^\Lambda = \times_{k \in \Lambda} M_k$,

$$d\mu_\Lambda(x_\Lambda) = \frac{\exp\left\{-\lambda \sum_{(k,j) \in \Lambda} V(x_k, x_j)\right\}}{\int_{M^\Lambda} \exp\left\{-\lambda \sum_{(k,j) \in \Lambda} V(x_k, x_j)\right\} \prod_{k \in \Lambda} d\sigma(x_k)} \times d\sigma(x_k).$$

Here, the sum is taken over $k, j \in \Lambda$ such that $|k-j| = 1$, $V \in C^\infty(M \times M)$, and $\sigma$ denotes a Riemann volume normalized to one. For sufficiently small $\lambda$ [4], the family $\{\mu_\Lambda\}$ has a probability measure $\mu$ on the Borel $\sigma$-algebra on $M^{\mathbb{Z}^d}$ as the only weak thermodynamic limit for $\lambda \nearrow \mathbb{Z}^d$.

The Dirichlet form of the measure $\mu$ is determined by the relation

$$\frac{1}{2} \int_{M^{\mathbb{Z}^d}} \sum_{k \in \mathbb{Z}^d} \langle \nabla_k u, \nabla_k v \rangle d\mu$$

on the set of smooth cylindrical functions $u, v \in C^\infty_{cyl}(M^{\mathbb{Z}^d})$; here, $\nabla_k$ denotes the operator of covariant differentiation on the manifold $M_k$. On $C^\infty_{cyl}(M^{\mathbb{Z}^d})$, the Dirichlet operator $H_\mu$ corresponding to this form has the representation

$$H_\mu = \sum_{k \in \mathbb{Z}^d} H_{\mu,k}.$$
where $H_k = -\Delta_k/2 - \lambda \langle b_k, \nabla_k \rangle / 2$ with the logarithmic derivative

$$b_k = -\nabla_k \left( \sum_{|k-j|=1} V(x_k, x_j) \right).$$

It is clear that every operator $H_{\mu,k}$ is Hermitian in the space $L_2(M^{\mathbb{Z}^d}, \mu)$ with domain $C^\infty_{cyl}(M^{\mathbb{Z}^d})$. Below, we show the conditions of essential self-adjointness of the operator $H_\mu$ on the domain $C^\infty_{cyl}(M^{\mathbb{Z}^d})$, i.e., the conditions for the relation $H_\mu^* = \hat{H}_\mu$ to be true, where $\hat{H}_\mu$ denotes the closure $H_\mu$ in $L_2(M^{\mathbb{Z}^d}, \mu)$. We prove that the corresponding semigroup preserves a certain class of smooth functions $E \subset \mathcal{D}(H_\mu)$, which provides the essential self-adjointness of the operator $H_\mu$ [5].

Fix $B_a = [1, a]^d \cap \mathbb{Z}^d$ for sufficiently large $a \in \mathbb{N}$ and consider a set

$$U(0) \subset \mathbb{Z}^d, \quad U(0) = \bigcup_{k \in (2aZ)^d} \theta_k B_a,$$

where $\theta_k$ is a shift by a vector $k \in \mathbb{Z}^d$. Put $U(i) = \theta(i_1,0,...,0) \theta(0,0,...,i_d) U(0)$, where the collection $i = (i_1,...,i_d)$, $i_s \in \{0,1\}$. The sets $U(i)$ determine the partition of the lattice $\mathbb{Z}^d$ into infinite subsets which do not intersect. This partition is associated with the representation of the operator $H_\mu$ as an infinite sum of operators $H(i)$ on $C^\infty_{cyl}(M^{\mathbb{Z}^d})$:

$$H_\mu = \sum_{i \in \{0,1\}^d} H(i),$$

where

$$H(i) = \sum_{k \in U(i)} H_{\mu,k}.$$

Due to the special block structure of the operator $H(i)$, the Cauchy problem

$$\frac{df(t,x)}{dt} = -H(i)f(t,x),$$

$$f(0,x) = f_0(x) \in C^\infty_{cyl}(M^{\mathbb{Z}^d})$$

(1)

is smoothly solvable in the space $C^\infty_{cyl}(M^{\mathbb{Z}^d})$, but the support of the cylindricity of the function $f(t,x)$ can widen in comparison with the support of the function $f_0(x)$.

The smooth solvability of problem (1) in the space $C^\infty_{cyl}(M^{\mathbb{Z}^d})$ also guarantees the smooth solvability of Cauchy problem (1) with operator $\hat{H}(i)$, where $\hat{H}(i)$ denotes a closure of $H(i)$ in the space $L_2(M^{\mathbb{Z}^d}, \mu)$. Taking into account that each of the operators $H(i)$ is positive in $L_2(M^{\mathbb{Z}^d}, \mu)$, we conclude that the operators $H(i)$ are essentially self-adjoint in $L_2(M^{\mathbb{Z}^d}, \mu)$ with essential domain $C^\infty_{cyl}(M^{\mathbb{Z}^d})$.

Thus, the problem on essential self-adjointness of the operator $H_\mu$ turns into the problem on essential self-ad-