By using a new method suggested in the first part of the present work, we study systems which become linear in the zero approximation and have perturbations in the form of polynomials. This class of systems has numerous applications. The following fact is even more important: Our technique demonstrates how to generalize the classical method of Poincaré-Birkhoff normal forms and obtain new results by using group-theoretic methods. After a short exposition of the general theory of the method of asymptotic decomposition, we illustrate the new normalization technique as applied to models based on the Lotka-Volterra equations.

1. Asymptotic Decomposition in the Space of Homogeneous Polynomials

1.1. Transition to a Matrix with Simple Structure. Consider a system of almost linear ordinary differential equations

\[ \dot{x} = A x + \varepsilon \bar{o}(x') \quad x'(t_0) = x_0, \quad (1) \]

where \( \bar{o}(x') \) is a column vector whose elements are polynomials in \( n \) variables \( x_1, \ldots, x_n \) of degrees not greater than \( k \).

Assume that the matrix \( A \) of coefficients of the system of zero approximation corresponding to system (1) is semisimple (e.g., diagonalizable). This restriction imposed on \( A \) is not essential because any \( n \)-dimensional system (1) with nondiagonalizable matrix can be transformed into an \( (n + 1) \)-dimensional system with diagonalizable matrix. For the required transformation, the matrix \( A \) must be represented as a sum of diagonalizable and nilpotent components [2, p. 112], namely,

\[ A = A_d + A_n, \quad A_d A_n \equiv A_n A_d, \quad A_n^k \equiv 0, \quad k \leq n, \]

by using one of the well-known techniques. By substituting these identities into the original system, after the change of variables

\[ x' = \exp A_n(t-t_0) y', \quad x'(t_0) = x_0, \]

we obtain the system \( y' = A_d y' + \varepsilon \bar{\Omega}(t, y') \), where \( \bar{\Omega}(y') = \exp (-A_n(t-t_0)) \bar{o}(\exp A_n(t-t_0) y') \) is a vector of homogeneous polynomials of finite degree. By introducing the additional variable \( y_{n+1}' = t, \ y_{n+1} = 1 \), we arrive at a system of \( n + 1 \) equations with diagonalizable matrix.

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1.2. General Settings. Parallel with the linear space $V$ over $P$ generated by the elements $x_1, \ldots, x_n$ we consider the linear space $V_{\otimes v}$ over $P$ defined as the direct product of $v$ spaces $V$, i.e.,

$$V_{\otimes v} = \bigotimes_{\overset{\text{v}}{\mathbb{V}}} V.$$

A basis in the space $V_{\otimes v}$ is generated by all possible monomials of the form $x_1^{m_1} \ldots x_n^{m_n}, m_1 + \ldots + m_v = v$. Its dimensionality is denoted by $m_v$. The row vector composed of the elements of the basis in $V_{\otimes v}$ is denoted by $\mathbf{\hat{x}}_{m_v}$. It is evident that $m_1 = n$ and $\mathbf{\hat{x}}_{m_1} = \|x_1, \ldots, x_n\|_1$.

By $T(V)$ we denote an infinite-dimensional space defined as the direct sum of the subspaces of $V_{\otimes v}$, i.e.,

$$T(V) = \bigoplus_{\overset{\text{v}}{\mathbb{V}}} (V \otimes (V \otimes V) \otimes \ldots \otimes (V \otimes \ldots \otimes V)) \otimes \ldots.$$

The space $T(V)$ is closely connected with an infinite-dimensional Lie algebra. Let us describe its structure.

Let $Q$ be a constant $m_v \times n$ matrix with elements $q_{ij} \in P$, where $i = 1, m_v, j = 1, n$, and $q_1, \ldots, q_n$ are the elements of the row in the equality $q = \mathbf{\hat{x}}_{m_v} Q_\mathbf{q} \defeq \|q_1, \ldots, q_n\|_1$.

For an arbitrary sequence of matrices $Q$, the collection of differential operators

$$X = q_1 \frac{\partial}{\partial x_1} + \ldots + q_n \frac{\partial}{\partial x_n}, \quad q_i \in V_{\otimes v},$$

generates a linear space over $P$ denoted by $\mathcal{B}(V_{\otimes v})$. The matrix $Q$ is called the matrix of the operator $X$. One can easily show that the operator $X$ maps all vectors from the subspace $V_{\otimes i}$ into the subspace $V_{\otimes i+v-1}$.

By acting by the operator $X$ upon the basis $V_{\otimes i}$ we obtain

$$X \mathbf{\hat{x}}_{m_i} = \mathbf{\hat{x}}_{m_i+v-1} \mathcal{F}_{i+v-1}, \quad (2)$$

where $\mathcal{F}_{i+v-1}$ is an $m_i+v-1 \times m_v$ rectangular matrix. The matrix $\mathcal{F}_{i+v-1}$ of the operator $X$ which maps the space $\mathcal{B}(V_{\otimes i})$ into the space $\mathcal{B}(V_{\otimes v})$ and is defined by relation (2) is called the representation matrix of the operator $X$ in the space $V_{\otimes i}$. Let $Y$ be the differential operator of $\mathcal{B}(V_{\otimes v})$, namely,

$$Y = h_1 \frac{\partial}{\partial x_1} + \ldots + h_n \frac{\partial}{\partial x_n}, \quad h_i \in V_{\otimes v},$$

$$h = \mathbf{\hat{x}}_{m_v} \mathcal{H}, \quad h = \|h_1, \ldots, h_n\|, \quad \mathcal{H} \in R^{(m_v \times n)}.$$

It is easy to prove the following statement:

**Lemma 1.** Let the operators $X$ and $Y$ belong to the subspaces $\mathcal{B}(V_{\otimes v})$ and $\mathcal{B}(V_{\otimes v})$. Let $Q$ and $\mathcal{H}$ be the matrices of the representations of these operators in the subspace $V$. Then the Poisson bracket

$$[X, Y] = c_1 \frac{\partial}{\partial x_1} + \ldots + c_n \frac{\partial}{\partial x_n}, \quad c = \mathbf{\hat{x}}_{m_v+v-1} C, \quad c = \|c_1, \ldots, c_n\|_1,$$