ON THE UNIQUENESS OF SOLUTIONS OF THE DIRICHLET
AND NEUMANN PROBLEMS FOR AN ELLIPTIC SECOND-ORDER
DIFFERENTIAL EQUATION ON A SEMIAxis

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For an elliptic second-order differential equation in a Banach space, we give a description of all solutions of the homogeneous Dirichlet and Neumann problems and establish conditions under which these problems are uniquely solvable.

1. Consider an equation

\[ y''(t) - Ay(t) = 0, \quad t > 0, \]

(1)

where \( A \) is a positive operator in a Banach space \( \mathcal{B} \). This means that \( \text{Dom}(A) = \mathcal{B} \), where \( \text{Dom}(\cdot) \) is the domain of definition of this operator, the resolvent set of \( A \) contains \( (-\infty, 0] \), and the following inequality is satisfied:

\[ \| (A + \lambda I)^{-1} \| \leq \frac{M}{1 + \lambda}, \quad \lambda > 0, \quad 0 < M = \text{const.} \]

(2)

For positive \( A \), one can indicate \( 0 < \varphi < \pi \) and \( M(\varphi) > 0 \) such that estimate (2) with \( M(\varphi) \) and \( |\lambda| \) instead of \( M \) and \( \lambda \), respectively, is true in the sector \( \{ \lambda : \varphi \leq |\arg \lambda| \leq \pi \} \) [1]. Denote by \( \omega \) the greatest lower bound of these \( \varphi \). The pair \( (\omega, M) \) is called the type of the operator \( A \). If \( A \) is positive, then one can define its fractional power \( A^\alpha \), \( 0 \leq \alpha \leq 1 \), which is also a positive operator of type \( (\alpha \omega, M) \). Under the condition \( \omega < \pi/2 \), the operator \( -A \) generates a bounded holomorphic semigroup of linear operators on \( \mathcal{B} \) [1, 2]. This means that \( -\sqrt{A} \) is the generator of a bounded holomorphic semigroup. Denote this semigroup by \( U(t) \). We set

\[ U(t) f = \lim_{\tau \to 0+} \text{ind}_{\tau} \mathcal{B} \cdot (-\sqrt{A}), \]

where \( \mathcal{B} \cdot (-\sqrt{A}) \) is the complement of \( \mathcal{B} \) in the norm \( \| f \|_{-\tau} = \| (U(t) f) \| \), \( f \in \mathcal{B} \) (\( \| \cdot \| \) is the norm in \( \mathcal{B} \)). As shown in [3], the semigroup \( U(t) \) can be extended to an equicontinuous \( C_0 \)-class semigroup \( \hat{U}(t) \) on a locally convex topological space \( \mathcal{B} \cdot (-\sqrt{A}) \). The extension of the operator \( -\sqrt{A} \) to the space \( \mathcal{B} \cdot (-\sqrt{A}) \) by continuity is its generator. If \( t > 0 \), \( f \in \mathcal{B} \cdot (-\sqrt{A}) \), then \( \hat{U}(t) f \in \text{Dom}(A) \).

Let \( B \) (\( \text{Dom}(B) = \mathcal{B} \)) be a closed linear operator in \( \mathcal{B} \), and let

\[ C_\alpha(n!)(B) = \left\{ f \in \bigcap_{n \in N_0} \text{Dom}(B^n) : \exists c > 0 : \| B^k f \| \leq c \alpha^k k!, \quad \forall k \in N_0 \right\}, \]

where \( \alpha > 0 \) and \( N_0 = \{ 0, 1, 2, \ldots \} \). The space \( C_\alpha(n!)(B) \) is a Banach space with respect to the norm...
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\[ \|f\|_{C_\alpha(n!)(B)} = \sup_{n \in \mathbb{N}_0} \left\| B^n f \right\|_{\alpha^n n!}. \]

We equip the sets

\[ C_{\{n!\}}(B) = \bigcup_{\alpha > 0} C_\alpha(n!)(B) \quad \text{and} \quad C_{(n!)}(B) = \bigcap_{\alpha > 0} C_\alpha(n!)(B), \]

respectively, with the topologies of inductive and projective limit of the Banach spaces \( C_\alpha(n!)(B) \), i.e.,

\[ C_{\{n!\}}(B) = \lim \text{ind} \ C_\alpha(n!)(B) \quad \text{and} \quad C_{(n!)}(B) = \lim \text{pr} \ C_\alpha(n!)(B). \]

Elements of the space \( C_{\{n!\}}(B)(C_{(n!)}(B)) \) are called analytic (entire) vectors of the operator \( B \). If \( A \) is a positive operator, then (see, e.g., \[4\])

\[ \mathcal{R}(\sqrt{A}) = \bigcup_{t > 0} \mathcal{R}(U(t)) \quad \text{and} \quad C_{\{n!\}}(\sqrt{A}) = \bigcap_{t > 0} \mathcal{R}(U(t)), \]

where \( \mathcal{R}(\cdot) \) is the range of values of the operator. Moreover, these spaces are compact in \( \mathcal{B} \) and invariant under the action of \( U(t) \). Note, that in the case where \( A \) is a positive self-adjoint operator in a Hilbert space, the space \( \mathcal{B}_{\{n!\}}(\sqrt{A}) \) coincides with the space dual to the space of analytic vectors of the operator \( \sqrt{A} \).

2. We say that a function \( y(t): (0, \infty) \to \mathcal{D}(A) \) is a solution of Eq. (1) on \( (0, \infty) \) if it is twice strongly continuously differentiable and satisfies Eq. (1). It should be emphasized that we impose no restrictions on the behavior of solutions in the vicinity of the origin. In \[3\], it was shown that each solution of Eq. (1) on \( (0, \infty) \) approaches the limiting value as \( t \to 0_+ \) in the space \( \mathcal{B}_{\{n!\}}(\sqrt{A}) \).

We say that the homogeneous Dirichlet problem is posed for Eq. (1) if it is necessary to find its solution \( y(t) \) on \( (0, \infty) \) such that

\[ \lim_{t \to 0_+} y(t) = 0 \]

[the limit is taken in the topology of the space \( \mathcal{B}_{\{n!\}}(\sqrt{A}) \)]. We define \( \cosh \sqrt{A} t \) and \( \sinh \sqrt{A} t / \sqrt{A} \) by the formulas

\[ \cosh \sqrt{A} t f = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^k f, \quad f \in C_{(n!)}(\sqrt{A}); \]

\[ \frac{\sinh \sqrt{A} t}{\sqrt{A}} f = \int_0^t \cosh \sqrt{A} s f ds = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A^k f, \quad f \in C_{(n!)}(\sqrt{A}). \]

It is easy to see that \( \cosh \sqrt{A} t f \) and \( (\sinh \sqrt{A} t / \sqrt{A}) f \) from \( f \in C_{(n!)}(\sqrt{A}) \) are entire vector functions with values in the space \( C_{(n!)}(\sqrt{A}) \).