shown that the statistics $\kappa_{i,q}$ and $\chi_{i,q}$ introduced in this paper are indeed measures of
closeness between samples belonging to the class of more refined differential measure of
closeness than the coarse metric (22). In this connection, the test criterion constructed
above is recommended for identification of distribution functions in mixture models

$$F_2(u) = (1 - \alpha) F_1(u) + \alpha \Phi(u).$$

We now proceed to the problem of constructing the interval $\mathcal{I}_{i,q}$, which is the basis
of the tests proposed in this paper. Evidently, such an interval must be selected on the
region $(a, b)$ of the line where the distinction between the probability densities $f_1(u)$ and
$f_2(u)$ (for example in the metric $C$ or $\mathcal{L}_1$) is the largest. Since these densities are un-
known, they should be replaced by the histograms $f_1^*(u)$ and $f_2^*(u)$ constructed with the aid
of the well-known methods of histogram estimation based on learning samples $x_1$ and $x_2$, re-
spectively. Usually it is not too difficult to determine, using the graphs of these histo-
grams, the interval in which $f_1^*(u)$ and $f_2^*(u)$ differ to the largest extent: Having this
interval $(a, b)$, one could obtain order statistics $x_1(1)$ and $x_1(1+q)$ constructed by means
of the sample $x_1$ which contain the interval $(a, b)$ or form an interval $\mathcal{I}_{i,q}$ that differs
only slightly from the interval $(a, b)$.

LITERATURE CITED

1. S. A. Matveichuk and Yu. I. Petunin, "A generalization of the Bernoulli model occurring
   25, 762-768 (1954).
4. L. N. Bol'shev and N. V. Smirnov, Tables of Mathematical Statistics [in Russian],
   (1946).
6. G. M. Fikhtengol'ts, Course of Differential and Integral Calculus, 3 vols. [in Russian],
   Nauka, Moscow (1966).

GENERALIZED KILLING TENSORS OF ARBITRARY RANK AND ORDER

A. G. Nikitin

We define Killing tensors and conformal Killing tensors of arbitrary rank and
order which generalize in a natural way the notion of a Killing vector. We
explicitly derive the corresponding tensors for a flat de Sitter space of di-
ension $p + q = m$, $m \leq 4$, which permits us to calculate complete sets of sym-
metry operators of arbitrary order $n$ for a scalar wave equation with $m$ indepen-
dent parameters.

1. Introduction. In recent years the classical group-theoretical approach [1] has
been increasingly replaced with more modern methods for studying the symmetry of differen-
tial equations. In particular, more attention has been placed on the study of symmetry
operators of higher orders which are a natural generalization of generators of Lie groups
and which contain important information on the hidden symmetry of the equation. These
operators are used to describe systems of coordinates in which the equation can be solved
by a separation of variables [1-4], in the study of non-Noetherian conservation laws, etc.
[5].

Institute of Mathematics, Academy of Sciences of the Ukrainian SSR, Kiev. Translated
from Ukrainskii Matematicheskii Zhurnal, Vol. 43, No. 6, pp. 786-795, June, 1991. Original
article submitted October 4, 1990.
It is known that the description of first-order symmetry operators (generators of Lie groups) is based on the explicit calculation of the Killing vector \([6, 7]\) corresponding to the space of independent variables. Symmetry operators of higher orders have more complicated structures associated with them, which are called Killing tensors (or conformal Killing tensors) of rank \(j\) and order \(s\) \((j, s = 1, 2, \ldots)\).

In this article we define these tensors as a complete set of linearly independent solutions of some overdetermined system of partial differential equations and compute them explicitly for all cases where the number of independent variables \(m\) is less than or equal to four.

The obtained results can be used to study the higher symmetries of a large class of equations of mathematical physics in \(m\) independent variables. As an example (which is of independent interest), in this article we describe a complete set of symmetry operators of arbitrary order \(n\) for the scalar wave equation in an \(m\)-dimensional space.

2. Symmetry Operators for a Wave Equation. To arrive at a natural definition of Killing tensors of arbitrary rank and order, we state the problem of determining symmetry operators of arbitrary order \(n = 1, 2, \ldots\) for a wave equation

\[
L\varphi = (g_{\mu\nu}\partial^\mu\partial^\nu - \kappa^2)\varphi = 0, \quad \partial^\mu = \frac{\partial}{\partial x^\mu},
\]

where \(\kappa\) is a real parameter, \(g_{\mu\nu}\) a metric tensor whose non-zero elements are equal to \(g_{00} = -g_{11} = -g_{22} = \ldots = 1,\ \mu, \nu = 0, 1, \ldots, m - 1, m \leq 4\), and repeated indices mean summation.

For our purposes, it suffices to study only solutions of Eqs. (1) which are defined on some open set \(D\) of an \(m\)-dimensional manifold \(\mathbb{R}^m\) consisting of points with coordinates \((x_0, x_1, \ldots, x_{m+1})\), and which are analytic with respect to the real parameters \(x_0, \ldots, x_{m+1}\). The space of solutions of Eq. (1) for a fixed \(D\) is denoted by \(\mathcal{F}_\varphi\).

Let \(\mathcal{F}\) be the vector space of all complex-valued functions defined on \(D\) which are real-analytic, and let \(L\) be a linear differential operator (1) defined on \(\mathcal{F}\). Then \(L\varphi \in \mathcal{F}\) if \(\varphi \in \mathcal{F}\) and \(\mathcal{F}_\varphi\) is the nullspace (kernel) of the operator \(L\).

Let \(\mathcal{M}_n\) be the set (class) of linear differential operators of order \(n\) defined on \(\mathcal{F}\). Then the symmetry operator \(Q \in \mathcal{M}_n\) of Eq. (1) is defined as follows.

**Definition 1.** A linear differential operator \(Q\) of order \(n\) defined by

\[
Q = \sum_{i=0}^{n} h_{\alpha_1\cdots\alpha_i} \partial^{\alpha_1} \cdots \partial^{\alpha_i}, \quad h_{\alpha_1\cdots\alpha_i} \in \mathcal{F},
\]

is called a symmetry operator of Eq. (1) in class \(\mathcal{M}_n\) (or a symmetry operator of order \(n\) if

\[
\{Q, L\} = \alpha_0 L, \quad \alpha_0 \in \mathcal{M}_{n-1},
\]

where \(\{Q, L\} = QL - LQ\) is the commutator of the operators \(L\) and \(Q\).

In the case \(n = 1\) the symmetry operators defined above can be regarded as generators of the invariance group of Eq. (1). Symmetry operators of order \(n > 1\) do not generate a Lie group and instead define a generalized (non-Lie) symmetry. The problem of describing a complete set of symmetry operators of order \(n\) for Eq. (1) reduces to finding a general solution of operator equations (3).

3. Killing Tensors of Rank \(j\) and Order \(s\). It is convenient to write all operators appearing in Eq. (3) as sums of \(j\)-multiple anticommutators

\[
Q = \sum_{j=0}^{n} \hat{Q}_j, \quad \alpha_j = \sum_{i=0}^{n-j} \hat{\alpha}_j, \quad \alpha_j L = \frac{1}{4} \{[\alpha_0, \partial_{\alpha_0}], \partial_{\alpha_0}\} + \frac{1}{2} \{[\partial^\mu \alpha_0], \partial_{\mu}\},
\]

where

\[
\hat{Q}_j = [\ldots [F^{a_1\cdots a_j}, \partial_{a_1}], \partial_{a_2}], \ldots, \partial_{a_j}],
\]

\[
\hat{\alpha}_j = [\ldots [\alpha^{a_1\cdots a_j}, \partial_{a_1}], \partial_{a_2}], \ldots, \partial_{a_j}],
\]

735