A CLASS OF SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS

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Exact solutions of the boundary layer equations can be obtained in closed form only in rare cases. These generally involve self-similar solutions for which the corresponding ordinary differential equation can be integrated exactly. In this paper solutions of more general form, containing additive functions of the longitudinal x coordinate in the expressions for the stream function and the self-similar variable, are considered in relation to two-dimensional steady boundary layers. This makes it possible to enlarge the class of problems whose solutions are analytic expressions and in a number of cases can be obtained in the form of expressions containing arbitrary functions of x, which makes possible various interpretations of the solution. In order to introduce arbitrary functions into the solutions of the equations of the axisymmetric boundary layer the problem is reduced to an overdetermined system of ordinary differential equations. This method is also capable of being applied more widely.

1. We will seek the solutions of the equations of a plane steady incompressible boundary layer in the form:

$$\psi = \alpha(x) + \mu(x)\phi(\xi), \quad \xi = -\frac{y}{\beta(x)} + q(x), \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$  \hspace{1cm} (1.1)

Here, \(\psi\) is the stream function, and \(u\) and \(v\) are the longitudinal and transverse velocity components. Substituting (1.1) in the boundary layer equations, we arrive at the following relation:

$$\mu'\beta(\phi'' - q\phi''') - \mu\beta'\phi'' = R(x) + v\psi'''$$  \hspace{1cm} (1.2)

$$R = UU'\beta^3\mu^{-1}.$$  

Here, \(U(x)\) is the velocity of the external potential flow.

Requiring that (1.2) reduces to an ordinary differential equation for \(\phi(\xi)\), we obtain

$$\mu'\beta = \text{const}, \quad \mu\beta' = \text{const}, \quad \alpha'\beta = \text{const}, \quad R = \text{const}$$  \hspace{1cm} (1.3)

These relations impose fairly severe limitations on the form of the functions \(\mu, \beta, \alpha,\) and \(U\). For solutions with \(\alpha = 0\) this requirement leads to power-law or exponential relations \(\mu(x), \beta(x),\) and \(U(x)\) [1].

The function \(q(x)\) does not enter into (1.2) and (1.3) and therefore can remain arbitrary. Thus, choosing an expression for the variable \(\xi\) in the form (1.1) makes it possible to introduce into the solution an arbitrary function of the longitudinal coordinate.

Let us consider the further possibilities that result from choosing the expression for \(\psi\) in the form (1.1). In general, the introduction of the function \(\alpha(x)\) into the solution is equivalent to the overdetermination of \(\phi(\xi)\), since from (1.3) there follows \(\alpha = c_1\mu + c_2\) (\(c_1\) and \(c_2\) are constants). The two special cases considered below are exceptions.

The first of these corresponds to \( \mu = \text{const} \). Assuming, without loss of generality, that \( \mu = 1 \), from (1.3) we obtain

\[
\alpha = a \ln x, \quad \beta = x, \quad U = c/x \quad (1.4)
\]

\[
f'' + kf' + f^3 - 1 = 0, \quad k = a/\sqrt{\eta c}, \quad f = \sqrt{\eta c}, \quad \eta = \sqrt{\eta c} / \sqrt{\xi}
\]

(1.5)

where \( a \) and \( c \) are constants.

For a certain relation between the quantities \( a, c, \) and \( \nu \) the solution of Eq. (1.5) can be obtained in quadratures. Making the change of variables

\[
f' = z^*w(z) + 1, \quad z = e^z, \quad z^*w'' + (2n + 1 + k) zw' + (n^2 + kn + 2) w + z^*w^3 = 0
\]

in (1.5), we find that in the case

\[
k = \pm 5/\sqrt{3}, \quad n = \mp 2/\sqrt{3}
\]

(1.6)

(1.7)

the equation for \( w \) admits an integrating multiplier \( 2z - nzw' \) and, thus, reduces to the following:

\[
z^z - nzw^2 + zzw^3 + A = 0
\]

Hence, setting the constant of integration \( A \) equal to zero and taking (1.6) and (1.7) into account, we arrive at an expression for \( f' \) and, using (1.4), (1.7), (1.1), and (1.5), the following expressions for the velocity components:

\[
f' = 1 - 2[B \exp(\pm \eta / \sqrt{3} + 1)]^{-2}, \quad u = c/x[f'(\eta)], \quad v = 5 / \sqrt{3} \exp[-c/x]\]

(1.8)

In these expressions the upper sign corresponds to the case \( c > 0 \), and the lower sign to the case \( c < 0 \) \((U = -c/x^{-1})\).

In the second case, in which introducing the function \( \alpha(x) \) into the solution does not reduce to a trivial overdetermination, \( \varphi(\xi) \) corresponds to \( \mu = \text{const} \) and \( \beta = \text{const} \) (we assume that \( \mu = \beta = 1 \)), so that relations (1.3) give

\[
\alpha = ax, \quad U = c
\]

(1.9)

where \( a \) and \( c \) are constants. Then from (1.2) we obtain a linear equation for \( \varphi(\xi) \), whose solution leads, when (1.1) and (1.9) are taken into account, to the following expressions for \( \psi, u, \) and \( v \):

\[
\psi = ax + A \exp\left(-\frac{a z}{\nu}\right) + c z, \quad z = y + q(x), \quad u = c - \frac{a}{\nu} A \exp\left(-\frac{a z}{\nu}\right), \quad v = -a - uq'(x)
\]

(1.10)

The solutions (1.8) and (1.10) can be used for describing the boundary layers on permeable surfaces. Expressions (1.8) with \( q(x) = 0 \) describe the boundary layer for an external flow of the type \( U = c/x \) on a surface through which in the case \( c > 0 \) (divergent channel flow) fluid is sucked and in the case \( c < 0 \) (convergent channel flow) fluid is blown along the normal to the surface at a rate \( v_0 = b/x, \) where \( |b| = 5\sqrt{\eta c} / \sqrt{3} \). The constant \( B \) is determined from the condition \( f'(0) = 0 \).

Generally speaking, the longitudinal velocity distribution (1.8) satisfies the condition \( u \to U \) as \( x \to 0 \); however, the point \( x = 0 \) must be excluded from consideration (in the neighborhood of a source or sink the assumptions of boundary layer theory are not satisfied), which eliminates the question of the singularity in the expression for \( v \) at \( x = 0 \). A similar situation is encountered in the problem of the boundary layer on a flat plate with suction or blowing distributed in accordance with the law \( v_0 = b/x \) [1].

The solution (1.8) indicates the possibility of forming, as a result of sucking fluid through the wall, a separationless \((f''(0) > 0)\) boundary layer in a divergent channel flow in which normally separation always takes place.

For \( q(x) = 0 \) and a constant \( A \) determined by the condition \( u = 0 \) when \( \xi = 0 \) expressions (1.10) give the known solution of [2] (see also [1]) describing uniform flow over