IN Variant R-Linkals of Some Continuous Additive Operators

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We prove that an infinite-dimensional Banach space X contains a nontrivial closed R-lineal X₀ invariant both under the action of a compact additive operator A and under the action of all continuous additive operators T in X such that T = T₁ + T₂, where T₁A = AT₁ and T₂A = −AT₂. If A is a linear or antilinear compact operator, then X₀ is a subspace of X.

Let X be a complex Banach space. A continuous additive operator A is defined as a continuous mapping A : X → X satisfying the condition (∀x₁, x₂ ∈ X) : A(x₁ + x₂) = Ax₁ + Ax₂. A compact additive operator A is defined as an additive operator A : X → X which transforms any bounded set into a precompact set. A subset L ⊆ X is called an R-lineal if (∀α₁, α₂ ∈ R) (∀x₁, x₂ ∈ L) : α₁x₁ + α₂x₂ ∈ L. If (∀x ∈ L) : Ax ∈ L, then the R-lineal L is called invariant with respect to the continuous additive operator A.

In the present paper, we study the problem of existence of invariant R-lineals and subspaces for a compact additive operator A and continuous additive operators T connected with A by certain commutation relations. The results obtained generalize the theorems proved by Lomonosov in [1, p. 55] and by the author in [2, p. 104].

1. Preliminary Information (see also [3, pp.3–12, 4, pp.476–479, 5, pp.18–21])

Any continuous additive operator A acting in X is R-linear, i.e., [6, p.100]

(∀α₁, α₂ ∈ R) (∀x₁, x₂ ∈ X) : A(α₁x₁ + α₂x₂) = α₁Ax₁ + α₂Ax₂.

Hence, continuous additive operators possess the major part of algebraic and topological properties of continuous linear operators.

A continuous additive operator A is called a continuous antilinear operator if it satisfies the condition (∀α ∈ C) (∀x ∈ X) : A(αx) = ̅αAx. If a continuous antilinear operator J in X is such that J² = I (I is the identity operator in X) and (∀x ∈ X) : ||Jx|| = ||x||, then J is called an (isometric) involution. If the space X is a complex linear span of the R-lineal L ⊆ X such that (∀x ∈ L, x ≠ 0) : ix ∈ L, then L is called a real kernel of the space X. The collection of all fixed points x ∈ X of the involution J forms the real kernel of the space X or, in other words, the J-real kernel of this space. A continuous additive operator A commuting with the involution J is called J-real.

If, in a complex linear continuous space X, we discard the possibility of multiplying nonreal numbers by vectors, then X turns into a real linear continuous space X₉. A continuous additive operator A in X corresponds in the space X₉ to a continuous linear operator A₉ given by the equality (∀x ∈ X₉) : A₉x = Ax with x ∈ X on the right-hand side. Consider a complexification ̅X = X₉ + i ⊗ X₉ [4, p.477] of the space X₉ (the operation of multiplication of nonreal numbers by vectors in the space ̅X is denoted by symbol ✎ to distinguish it from the corresponding operation in X), where the norm of the space ̅X in the space X₉ coincides with the norm of the space X. Note that the spaces X and ̅X are complete. In the space ̅X, we define an operator ̅A by the formula
\((\forall x \in \bar{X}, x = x_1 + i \cdot x_2 (x_1, x_2 \in X_R)) : \tilde{A}x = A_Rx_1 + i \cdot A_Rx_2.\)

This operator is called a linear extension of the continuous additive operator \(A\) from the space \(X\) to the space \(\bar{X}\). One can prove that \(\tilde{A}\) is a continuous linear \(\bar{J}\)-real operator, where \(\bar{J}\) is an involution in \(\bar{X}\) with \(\bar{J}\)-real kernel \(X_R\).

If, for a number \(\lambda \in \mathbb{C}\), there exists a vector \(x \in X, x \neq 0, \) such that \(A(\lambda)x = 0, \) where \(A(\lambda) = A - \lambda I\) for \(\lambda \in \mathbb{R}\) and \(A(\lambda) = A^2 - 2 \text{Re} \lambda A + |\lambda|^2 I\) for \(\lambda \notin \mathbb{R}\), then we say that \(\lambda\) (or \(\bar{\lambda}\)) is an eigenvalue and \(x\) is an eigenvector of the continuous additive operator \(A\) corresponding to the number \(\lambda\) (or \(\bar{\lambda}\)). The collection of all eigenvectors of a continuous additive operator \(A\) in the space \(X\) corresponding to a given number \(\lambda\) complemented by zero forms an \(R\)-lineal \(L_\lambda\), which is called the characteristic \(R\)-lineal of the operator \(A\) corresponding to the number \(\lambda\) (or \(\bar{\lambda}\)).

**Lemma 1.** A number \(\lambda \in \mathbb{C}\) is an eigenvalue of a continuous additive operator \(A\) in \(X\) if and only if it is an eigenvalue of the linear extension \(\tilde{A}\) of the operator \(A\) to the space \(\bar{X}\). In this case, an \(R\)-lineal \(L_\lambda \subset X\) is a characteristic \(R\)-lineal of the continuous additive operator \(A\) in \(X\) corresponding to a number \(\lambda\) if and only if \(L_\lambda\) is a \(\bar{J}\)-real kernel of the subspace \(\tilde{L}_\lambda\) for \(\lambda \in \mathbb{R}\) and a \(\bar{J}\)-real kernel of the subspace \(\tilde{L}_\lambda \oplus \tilde{L}_{\bar{\lambda}}\) for \(\lambda \notin \mathbb{R}\); here, \(\tilde{L}_\lambda\) and \(\tilde{L}_{\bar{\lambda}}\) are the proper subspaces of the operator \(\tilde{A}\) corresponding to the numbers \(\lambda\) and \(\bar{\lambda}\), respectively.

The proof of Lemma 1 is based on the following fact: For a linear extension \(\tilde{B}\) of a continuous additive operator \(B\) from the space \(X\) to the space \(\bar{X}\), the \(R\)-lineal \(\text{Ker} B\) is a \(\bar{J}\)-real kernel of the subspace \(\text{Ker} \tilde{B}\). Further, note that \(\tilde{A} - \lambda I\) is a linear extension of the operator \(A - \lambda I (\lambda \in \mathbb{R})\) and

\[
\tilde{A}^2 - 2 \text{Re} \lambda \tilde{A} + |\lambda|^2 I = (\tilde{A} - \lambda I) (\tilde{A} - \bar{\lambda} I)
\]

is a linear extension of the operator \(A^2 - 2 \text{Re} \lambda A + |\lambda|^2 I (\lambda \in \mathbb{R})\). Therefore, \(L_\lambda = \text{Ker} (A - \lambda I)\) is a \(\bar{J}\)-real kernel of the subspace \(\tilde{L}_\lambda = \text{Ker} (\tilde{A} - \lambda I)\) for \(\lambda \in \mathbb{R}\) and \(L_\lambda = \text{Ker} (A^2 - 2 \text{Re} \lambda A + |\lambda|^2 I)\) is a \(\bar{J}\)-real kernel of the subspace \(\tilde{L} = \text{Ker} (\tilde{A}^2 - 2 \text{Re} \lambda \tilde{A} + |\lambda|^2 I)\) for \(\lambda \notin \mathbb{R}\). It is also worth noting that \(z^2 - 2 \text{Re} \lambda z + |\lambda|^2 = (z - \lambda)(z - \bar{\lambda})\) is the minimal annihilating polynomial of the operator \(\tilde{A}_0\) induced by the operator \(\tilde{A}\) in \(\tilde{L}\) and, hence,

\[
\tilde{L} = \text{Ker} (\tilde{A} - \lambda I) + \text{Ker} (\tilde{A} - \bar{\lambda} I) = \tilde{L}_\lambda + \tilde{L}_{\bar{\lambda}}.
\]

2. **Principal Theorems**

We prove Theorem 1 by using the Hilden approach to the proof of the Lomonosov theorem.

**Theorem 1.** Let \(A \neq 0\) be a compact additive operator in an infinite-dimensional Banach space \(X\) and let \(\mathcal{A}\) be the set of all continuous additive operators \(T\) in \(X\) each of which is representable as the sum of continuous additive operators \(T_1\) and \(T_2\) commuting and anticommuting with the operator \(A\), respectively, i.e.,

\[
\mathcal{A} = \{ T | T = T_1 + T_2, T_1A = AT_1, T_2A = -AT_2 \}.
\]

Then \(X\) contains a nontrivial closed \(R\)-lineal \(X_0\) invariant under the action of all operators \(T \in \mathcal{A}\).