Let us consider a bounded solution of system (1). In this case condition (3) is satisfied for \( a < 2s - n, s = \min s_i \). Therefore, for \( a < 2s - n \) we have the following corollary.

**COROLLARY.** Suppose that either \( 2s \leq n \) or

\[
K_i M_i^{a_i} (2s - n) < 1, \quad i = 1, N.
\]

Then every bounded solution of system (1) is a constant.

This result has already been derived in [4] in the case where \( s_i = t_i = 1, i = 1, N \).

**LITERATURE CITED**


**A DERIVATION OF EXACT ESTIMATES FOR THE DERIVATIVE OF THE SPLINE-INTERPOLATION ERROR**

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Using new facts on the behavior of derivatives in spline interpolation, we give a proof for exact estimates of the error in approximating the first derivative.

The problem of obtaining exact error estimates for simultaneous approximation of the derivative in spline interpolation is quite subtle; the known proofs, even for lower-order splines, are noted for their great complexity ([1], Sec. 5.1). New facts proven in [2] on the behavior of derivatives of the error in spline interpolation allow one to simplify considerably the considerations pertinent to the proof of the results presented in [3].

We consider the spaces \( C \) and \( L_p (1 \leq p < \infty) \) of \( 2\pi \)-periodic functions \( f(t) \) continuous on the entire axis and integrable on \((0, 2\pi)\) in the \( p \)-th power with the standard norm

\[
\| f \|_C = \max_{t} |f(t)|, \quad \| f \|_p = \left( \int_0^{2\pi} |f(t)|^p dt \right)^{1/p}.
\]

A priori information on the functions being interpolated is given by the class \( W_\infty^r \), \( r = 1, 2, \ldots \), of functions \( f(t) \in C^{r-1} \) for which the derivative \( f^{(r-1)}(t) \) is locally absolute continuous and \( \sup_{t \in \mathbb{R}} |f^{(r)}(t)| \leq 1 \).
As $S_{2n,m}$, $n, m = 1, 2, \ldots$, we shall denote the linear manifold of $2\pi$-periodic splines of order $m$ and defect 1 over a uniform partition $t_k = k\pi/n$, $k = 0, 1, \ldots, 2n$. It is known (see, for example, [1, p. 40]) that the linear manifold $S_{2n,m}$ interpolates at the points $\gamma_k = k\pi/n - [1 + (-1)^m] \pi/4n$, $k = 1, 2, \ldots, 2n$, i.e., at the zeros of the perfect spline $\Psi_{n,m+1}(t)$ defined by

$$\Psi_{n,0}(t) = \text{sgn} \sin nt, \quad \Psi_{n,v}(t) = \int_{\gamma_v}^t \Psi_{n,v-1}(u) \, du, \quad v = 1, 2, \ldots,$$

where $\gamma_v = 0$, if $v$ is even and $\gamma_v = \pi/2n$, if $v$ is odd. Consequently, for any function $f(t) \in C$ there exists a unique spline $s(f, t) \in S_{2n,m}$, satisfying the following conditions:

$$s(f, \gamma_k) = f(\gamma_k), \quad k = 1, 2, \ldots, 2n. \quad (1)$$

It is known [4; 5; 1, p. 194] that if $f(t) \in W_m^{m+1}$, then at every point $t$ the following exact inequality holds:

$$|f(t) - s(f, t)| \leq |\Psi_{n,m+1}(t)|, \quad (2)$$

from which for the difference $\delta(f, t) = f(t) - s(f, t)$ we immediately have the exact estimates

$$\|\delta(f)\|_c \leq \|\Psi_{n,m+1}\|_c,$$

$$\|\delta(t)\|_p \leq \|\Psi_{n,m+1}\|_p, \quad 1 \leq p < \infty. \quad (3)$$

To a function $f(t) \in C$ and a point $\theta = \theta_j = j\pi/2n$, $j = 1, 2, \ldots, 4n$, we shall relate the function $f_\theta(t)$ defined by

$$f_\theta(t) = \begin{cases} \frac{1}{2} [f(t) + f(2\pi - t)], & \text{if } |\Psi_{n,m+1}(0)| = \|\Psi_{n,m+1}\|_c, \\ \frac{1}{2} [f(t) - f(2\pi - t)], & \text{if } \Psi_{n,m+1}(0) = 0. \end{cases}$$

It follows that the graph of $f_\theta(t)$ as well as the graph of the spline $\Psi_{n,m+1}(t)$, is symmetric with respect to the straight line $t = \theta$ if $\theta$ is an extremum point of $\Psi_{n,m+1}(t)$, and is symmetric with respect to the point $\theta$, if $\Psi_{n,m+1}(\theta) = 0$. If $s(t) \in S_{2n,m}$, then also $s_\theta(t) \in S_{2n,m}$ for the function $s_\theta(t)$, similarly to $s(t)$, is $m - 1$ times continuously differentiable, while $s_\theta'(t)$ is piecewise continuous with possible discontinuities at the points $t_k = k\pi/n$. Obviously, $f(t) \in W_m^{m+1}$ implies $f_\theta(t) \in W_m^{m+1}$, and for the interpolating spline the relationship $s_\theta(f, t) = s(f, t)$ holds. We shall assume that $\delta(f, t) = f_\theta(t) - s_\theta(f, t)$.

Further considerations are based essentially on the following facts proven in [2].

**Proposition 1.** If $f(t) \in W_m^{m+1}$, then at any zero $\theta$ of the spline $\Psi_{n,m-v}(t)$, $v = 1, 2, \ldots, m - 2$, the following inequality holds:

$$|\delta_\theta(f, t)| \leq |\Psi_{n,m-v}(0)| = \|\Psi_{n,m-v+1}\|_c.$$

**Proposition 2.** Let $f(t) \in W_m^{m+1}, \Psi_\eta(f, t) = \Psi_{n,m+1}(t) - \delta(f, t)$. On the interval $(\alpha, \beta)$ between two consecutive zeros of the spline $\Psi_{n,m-v}(t)$ the derivative $\eta_\nu(f, t)$ changes its sign exactly once; in addition, it changes from minus to plus if $\Psi_{n,m-v}(t) > 0$ on $(\alpha, \beta)$ and from plus to minus if $\Psi_{n,m-v}(t) < 0$ on $(\alpha, \beta)$.

First we shall prove the following assertion.

**Theorem 1.** If $f(t) \in W_m^{m+1}$, then

$$\|\delta(f)\|_c \leq \|\Psi_{n,m-1}\|_c = \|\Psi_{n,m}\|_c. \quad (4)$$

$$\int_{\gamma_k}^{\gamma_{k+1}} |\delta(f, t)| \, dt \leq \int_{\gamma_k}^{\gamma_{k+1}} |\Psi_{n,m}(t)| \, dt = 4 \|\Psi_{n,m+1}\|_c, \quad k = 1, 2, \ldots, 2n. \quad (5)$$

The proof will be conducted in several steps.