We investigate the behavior of the best approximation of compressions of functions by trigonometric polynomials in the space $L_p, p < 1$.

1. In the theory of approximation of functions, numerous statements are proved which establish that if a function has a certain symmetry, then its best approximation in normed spaces is realized on elements with the same symmetry (see, e.g., [1]). For example, assume that a measurable function $f(x)$ from the normed space $L_p[-1, 1], 1 \leq p < \infty$, is either even or odd. It is known that, in this case, the best approximation of it by polynomials of given degree is realized on the set of even or odd polynomials, respectively.

Does this property of "preservation of symmetry" remain true in the case of approximation in nonnormable linear metric spaces?

Below, we give an example showing that, generally speaking, this is not true.

In the space $L_p[-1, 1], 0 < p < 1$, with the metric

$$
\rho(f, g)_p = \int_{-1}^{1} |f(x) - g(x)|^p dx,
$$

consider the best approximation of the function $f(x) = |x|, x \in [-1, 1]$, by algebraic polynomials of the first degree $Q_1(x) = ax + b$:

$$
E_1(|x|)_p = \inf_{Q_1} \rho(|x|, Q_1)_p.
$$

Assume that, for the approximation of this even function, it suffices to consider only even polynomials. However, an even polynomial of the first degree can only be a constant. Hence,

$$
E_1(|x|)_p = \inf_{\{Q_1 : Q_1(-x) = Q_1(x)\}} \rho(|x|, Q_1)_p = \inf_c \rho(|x|, c)_p =: E_0(|x|)_p.
$$

Let us calculate $E_0(|x|)_p$. First, note that it suffices to consider the case $c \in [0, 1]$. Then

$$
E_0(|x|)_p = 2 \inf_{0 \leq c \leq 1} \left( \int_0^c (c - x)^p dx + \int_c^1 (x - c)^p dx \right)
$$

$$
= 2 \inf_{0 \leq c \leq 1} (c^{p+1} + (1 - c)^{p+1}) = 2 \frac{1}{p+1} \left( \frac{1}{2^{p+1}} + \frac{1}{2^{p+1}} \right) = 2^{1-p} \frac{p}{p+1}.
$$
Thus, it follows from the assumption made above that

$$E_1(|x|)_p = E_0(|x|)_p = \frac{2^{1-p}}{p + 1}. \quad (1)$$

On the other hand, to estimate the quantity $E_1(|x|)_p$ from above, we take the odd polynomial $Q_1(x) = x$ as an approximating polynomial:

$$E_1(|x|)_p \leq \rho(|x|, x)_p = \frac{1}{2} \int_0^1 (2x)^p dx = \frac{2^p}{p + 1}. \quad (2)$$

However, in the case $p \in (0, 1/2)$, relations (1) and (2) contradict each other.

This difference between the properties of approximations in normed $L_p$-spaces, $p \geq 1$, and metric $L_p$-spaces, $p \in (0, 1)$, is explained by the fact that the corresponding averaging operators have unit norms for $p \geq 1$ and norms greater than one for $p < 1$. For example, it is easy to see that the operator $A$, $(Af)(x) = 2^{-1}(f(x) + f(-x))$, has the norms

$$\|A : L_p[-1, 1] \to L_p[-1, 1]\| = 1, \quad p \geq 1,$$

$$\|A : L_p[-1, 1] \to L_p[-1, 1]\| = 2^{1-p}, \quad p \in (0, 1).$$

2. In what follows, we assume that functions $f(x), x \in \mathbb{R}$, are measurable and $2\pi$-periodic. Consider their approximation in the metric of $L_p[-\pi, \pi], 0 < p < 1$, by trigonometric polynomials

$$T_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

of degree not higher than $n$:

$$E_n(f_p) := \inf_{T_n} \|f - T_n\|_p = \inf_{\{c_k\}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right|^p dx.$$

In the case of approximation in the normed spaces $L_p[-\pi, \pi], p \geq 1$, the following version of the property of "preservation of symmetry" is often useful: If the function $f$ is $2\pi/n$-periodic, then its polynomial of the best approximation is also $2\pi/n$-periodic. In particular, we have

$$E_{n-1}(f(n\cdot))_p = E_0(f(\cdot))_p, \quad p \geq 1. \quad (3)$$

This relation is useful for finding lower bounds for exact constants in the Jackson theorems [2].

Let us prove that, in the case of approximation in the metric spaces $L_p[-\pi, \pi], 0 < p < 1$, a certain weakened version of relation (3) remains true. Denote by $\Delta_s T(x)$ the difference of a function $T(x)$ with step $t$: $\Delta_s T(x) = T(x + t/2) - T(x - t/2)$.