ON PASSIVE AND ACTIVE ALGORITHMS OF RECONSTRUCTION OF FUNCTIONS

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We consider passive and active algorithms of reconstruction of functions, satisfying the condition
\[ |f(t') - f(t'')| \leq |t' - t''|^\alpha, \quad 0 < \alpha \leq 1, \]
according to their values \( f(t) \) at the points of the interval \([a, b]\).

An active algorithm is presented which guarantees, for monotonic functions from the above-mentioned class with \( 0 < \alpha < 1 \), a higher order of error in \( C[a, b] \) than can be attained by any passive algorithm.

1. The successes attained when solving extremal problems in the theory of approximations allow one to solve completely some problems of the optimal reconstruction of functions on the basis of \textit{a priori} and discrete information, as well (see, e.g., [1–3]). Moreover, in some cases the exact results follow immediately from the values of diameters. Nevertheless, the problem of the optimal reconstruction of the functional dependence on the basis of incomplete information has some peculiarities which should be considered independently.

The most important from the practical point of view is the case where a function \( f(t) \), given on a compact set \( I \), is reconstructed by its values at some points of this compact set. The problem becomes correct when we have some \textit{a priori} information about the function \( f(t) \) which allows us to include \( f(t) \) in some class of functions \( \mathcal{M} \) such that for any \( t_0 \in I \) the set \( \{ f : f \in \mathcal{M}, f(t_0) = 0 \} \) is bounded. Each set of points \( P_N = \{ t_1, t_2, \ldots, t_N \} \) from \( I \) is associated with a numerical vector
\[
f(P_N) = \{ f(t_1), f(t_2), \ldots, f(t_N) \},
\]
according to which the function \( \varphi(f(P_N), t) \) defined on \( I \) is constructed (for instance, an interpolating polynomial, or a spline) which approximately represents the function \( f(t) \). The reconstruction method, determined by the information vector (1) and the choice of a function \( \varphi(f(P_N), t) \), is denoted by \( (\varphi, f(P_N)) \). If \( X \) is a metric space of functions with distance \( \rho(f, g)_X \) containing both the class \( \mathcal{M} \) and the functions \( \varphi(f(P_N), t) \), for \( f \in \mathcal{M} \), then the value
\[
\rho(\mathcal{M}, (\varphi, f(P_N)))_X = \sup_{f \in \mathcal{M}} \rho(f, \varphi, f(P_N))_X
\]
gives an estimate of the error of reconstruction by the method \( (\varphi, f(P_N)) \) on the class \( \mathcal{M} \). The optimal reconstruction problem consists in finding the value (for fixed \( N \))
\[
\rho_N(\mathcal{M}, X) = \inf_{(\varphi, f(P_N))} \rho(\mathcal{M}, (\varphi, f(P_N)))_X
\]
and indicating the method \( (\varphi, f(P_N)) \), which allows one to realize this least lower bound. The next statement of the problem is actually equivalent to the former one:

For a given \( \varepsilon > 0 \), one should indicate the least natural number \( N \) and the method \( (\varphi, f(P_N)) \) such that
\[
\rho(\mathcal{M}, (\varphi, f(P_N)))_X = \rho_N(\mathcal{M}, X) \leq \varepsilon.
\]
This means calculating or estimating (for fixed \( N \)) the value
\[ \mu_{\varepsilon}(M, X) = \inf_{(\varphi, f(P_N))} \mu_{\varepsilon}(M, (\varphi, f(P_N)))_{X}, \]

where

\[ \mu_{\varepsilon}(M, (\varphi, f(P_N)))_{X} = \sup_{f \in M} \inf \{ N : N \in \mathbb{N}^+, r(f, \varphi, f(P_N))_{X} \leq \varepsilon \}. \quad (2) \]

Note that the choice of the points \( t_1, t_2, \ldots, t_N \) which is at our disposal can be realized by two principally different methods. We can either choose all \( N \) points \( t_k \) immediately and then calculate values \( f(t_k), k = 1, 2, \ldots, N \), or choose them step by step, taking into account the values \( f(t_v), v = 1, 2, \ldots, k-1 \), at already selected points, when choosing the next point. The first method is called the “passive” algorithm, while the second one, which consists, in fact, of two subsequent algorithms (mappings)

\[ A^1 : (\{t_v\}_{v=1}^{k-1}, \{f(t_v)\}_{v=1}^{k-1}) \rightarrow t_k, \]

\[ A^2 : (\{t_v\}_{v=1}^{k}, \{f(t_v)\}_{v=1}^{k-1}) \rightarrow f(t_k), \]

is called the “active (adapted)” algorithm or strategy.

The active reconstruction algorithm can be interpreted in terms of the games theory: The first player, aiming to guarantee the minimal error, chooses the point \( t_k \), by using the algorithm \( A^1 \), and the second player gives the value \( f(t_k) \). Here, two essentially different situations are possible. The first one arises when the value \( f(t_k) \) is chosen, on the basis of some experience or physical test (the “game with nature”). In this case, the active reconstruction algorithm is reduced to the strategy \( A^1 \) of the first player who intends to use the information \( (f(t_1), f(t_2), \ldots, f(t_{k-1})) \), when choosing \( t_k \) in the best possible way.

However, if it is necessary to estimate the error of reconstruction by the active algorithm on some class of functions \( M \), then the values \( f(t_k) \) should be also chosen, and they are at the disposal of the second player. When trying to get the estimate for any function \( f \in M \), one should rely at each step upon the worst case. And the game becomes antagonistic: the first player, trying to get the minimal possible error, chooses the point \( t_k \) according to algorithm \( A^1 \), while the second one, trying to get the maximal possible error, chooses the value \( f(t_k) \) (remaining inside the class \( M \)).

The possibilities of passive and active algorithms in the problems of optimal reconstruction were investigated by many authors (see, e.g., [4–6] with sufficiently complete bibliography). In particular, the fact of coincidence of the best guaranteed results obtained by passive and active algorithms was established in [4] for fairly broad functional classes \( M \).

The present paper is mainly devoted to the consideration of the situation when the active algorithm guarantees a higher order of the estimate (as \( N \to \infty \)) than any passive one.

2. Let \( C[a, b] \) be a space of functions continuous on the interval \( [a, b] \) with ordinary norm, and let \( H^\alpha[a, b], 0 < \alpha \leq 1 \), be a class of functions \( f(t) \in C[a, b] \) satisfying the Hölder condition

\[ |f(t') - f(t'')| \leq |t' - t''|^{\alpha}, \quad t', t'' \in [a, b]. \]

We shall write \( C \) and \( H^\alpha \) when this cannot cause any misunderstanding. Let \( P_N \) be a set of points \( t_1, \ldots, t_N \) on \( [a, b] \) and the values \( f(t_k) \) of the function \( f \in H^\alpha \) are determined. If \( g(t) \in H^\alpha \) and \( g(t_k) = f(t_k), k = 1, 2, \ldots, N \), then