Internal Approach to External Sets and Universes *

Part 1
Bounded set theory

Abstract. A problem which enthusiasts of IST, Nelson’s internal set theory, usually face is how to treat external sets in the internal universe which does not contain them directly. To solve this problem, we consider BST, bounded set theory, a modification of IST which is, briefly, a theory for the family of those IST sets which are members of standard sets.

We show that BST is strong enough to incorporate external sets in the internal universe in a way sufficient to develop the most advanced applications of nonstandard methods. In particular, we define in BST an enlargement of the BST universe which satisfies the axioms of HST, an external theory close to a theory introduced by Hrbáček.

HST includes Replacement and Saturation for all formulas but contradicts the Power Set and Choice axioms (either of them is incompatible with Replacement plus Saturation), therefore to get an external universe which satisfies all of ZFC minus Regularity one has to pay by a restriction of Saturation. We prove that HST admits a system of subuniverses which model ZFC (minus Regularity but with Power Set and Choice) and Saturation in a form restricted by a fixed but arbitrary standard cardinal.

Thus the proposed system of set theoretic foundations for nonstandard mathematics, based on the simple and natural axioms of the internal theory BST, provides the treatment of external sets sufficient to carry out elaborate external constructions.

This article † is the first in the series of three articles devoted to set theoretic

* Partially supported by AMS grants in 1993 and 1994 and DFG grant in 1994.
† The authors are pleased to mention useful conversations on the topic of the paper, personal and in written form, with I. van den Berg, M. Brinkman, F. and M. Diener, E. Gordon, A. Enayat, C. W. Henson, K. Hrbáček, H. J. Keisler, P. Loeb, W. A. J. Luxemburg, Y. Peraire, A. Prestel, V. Uspensky, M. Wolff, M. Yasugi, as well as to thank the organizers of the Oberwolfach (February 1994) and Marseille (July 1994) meetings in nonstandard analysis for the opportunity to give preliminary reports.

Presented by Robert Goldblatt; Received November 15, 1994; Revised March 30, 1995

foundations of nonstandard mathematics, to be published by *Studia Logica*. This research was accomplished as a single paper, too long, indeed, to be published in this Journal as a single paper.

The following Preface and Introduction present the problems which motivated this research, and the results and conclusions, related both to this first part and the two next parts.

**Preface**

Since Kreisel [19] initiated consideration of axiomatic systems for nonstandard analysis, several approaches to this matter have been suggested.

First of all, this is **RZ**, the theory of Robinson and Zakon [27] (see also Keisler [17]), which axiomatizes nonstandard extensions 1 of mathematical structures in the same sense as, say, the list of axioms for linearly ordered sets axiomatizes the class of all linearly ordered sets.

We consider, however, the other approach which intends to axiomatize the universe of all sets, “the universe of discourse” as it is called sometimes, in a nonstandard way rather than to describe nonstandard structures in the standard universe of **ZFC**. This approach also splits in two principal directions which we call here **internal** and **external**. Both of them have something in common: both assume that the “universe of discourse” includes a proper part, the **class of all standard sets**, which we denote by $\mathcal{S}$ and which is informally identified with the universe of all sets considered by “classical”, non–nonstandard mathematics. But they differ from each other in the answer to the question of how $\mathcal{S}$ relates to the “universe of discourse”.

The internal approach sees the “universe of discourse” as an elementary extension of $\mathcal{S}$ which obeys a certain amount of Saturation (called Idealization, or weak Saturation, see footnote 10), and such that, with respect to $\mathcal{S}$, no new bounded (= parts of sets) collections of standard sets can be defined. The universe of a theory of this type is usually called **internal universe**; we

---

1 We refer the reader to Henson and Keisler [8], Hurd and Loeb [11], Keisler, Kunen, Miller, and Leth [18], Lindström [20], Luxemburg [22], Stroyan and Bayod [28] on matters of nonstandard mathematical structures.