The article classifies simple modules over a generalized Weyl algebra of degree 1 with basis ring D that is a commutative domain with limited minimality condition.

Introduction

Definition 1 [1-3]. Let D be some ring, and let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be a family of commutative automorphisms of the ring D, i.e., \( \sigma_i \sigma_j = \sigma_j \sigma_i \), and let \( a = (a_1, \ldots, a_n) \) be the set of nonzero elements of the center \( Z(D) \) of the ring D, where \( \sigma_i(a_j) = a_j \) for all \( i \neq j \).

By the generalized Weyl algebra \( A = D(\sigma, a) \) (abbreviated GWA) of degree n with basis ring D we will mean the ring that is obtained by appending, to D, \( 2n \) symbols \( X_1^+, \ldots, X_n^+, X_1^-, \ldots, X_n^- \) that satisfy the following defining relations:

\[
X_i^- X_j^+ = a_i, \quad X_i^+ X_j^- = \sigma_i(a_j).
\]

\[
X_i^+ a = \sigma^{+1}(a) X_i^+, \quad a \in D.
\]

\[
[X_i^-, X_j^+] = [X_i^+, X_j^-] = 0 \quad \forall i \neq j.
\]

We call the sets \( a \) and \( \sigma \) the defining elements and automorphisms, respectively, of the generalized Weyl algebra \( A \).

In this paper we study simple (left) \( A \)-modules of a GWA of degree 1, \( A = D(\sigma, a) = D[X, Y; \sigma, a], X = X_1^+, Y = X_1^-, \) where D is a commutative domain with limited minimality condition (lmc), i.e., for any ideal \( \mathfrak{p} \neq 0 \subset D \) the \( \mathfrak{p} \)-module \( D/\mathfrak{p} \) has finite length. Beginning with Sec. 2 (resp. Sec. 3), a defining automorphism \( \sigma \) of the ring D satisfies the following condition \( C \): for any maximal ideal \( \mathfrak{p} \) of the ring D and any integer \( n \neq 0 \), we have \( \sigma^q(\mathfrak{p}) \neq \mathfrak{p} \) (resp. \( D \) is a Dedekind domain). We have the following examples of such algebras: the Weyl algebra \( A_1 = \mathcal{A}_1(K) \) over a field \( K \), the factor-algebra \( U/(C - X) \) of the universal enveloping algebra \( U = \mathfrak{U}e_{\mathfrak{l}}(2,K) \) of the Lie algebra \( \mathfrak{sl}(2) \) with respect to the center \( C \) is a Casimir element, \( \lambda \in K \), and their quantum analogs and certain natural deformations [2-4].

As in [5, 6, 3, 4], we split the problem of classification (up to indecomposable elements of some Euclidean ring) of simple modules over a GWA \( A \) of degree 1 into two parts. The first (easy) part concerns simple \( A \)-modules with D-torsion (Theorem 1; in the theory of Lie-algebra representation, such modules are called weight modules), while the second (hard) part is concerned with simple \( A \)-modules without D-torsion (Theorem 5). In Sec. 2 we show (Theorem 2) that the Krull dimension of the algebra \( A \) is equal to 1. In Sec. 4 we develop techniques that can be used to describe simple modules over “quantum” algebras \( A_\hbar \). The last section is devoted to construction of a series of simple \( A \)-modules and indecomposable elements of Euclidean rings.

1. Classification of Simple \( A \)-Modules with D-Torsion.

1.1. We begin by presenting some basic definitions and facts concerning localization of noncommutative rings [6, 7]. Let \( A \) be a ring with 1, and let \( S \subset A \) be a multiplicatively closed set containing 1. By the (left) localization of the ring \( A \) with respect to \( S \) we mean the ring \( B = S^{-1}A \) that contains \( A \) as a subring and is such that any element of \( S \) has an inverse in \( B \) and \( B = \{ s^{-1}a \mid s \in S, a \in A \} \). The ring \( B \) (if it exists) is unique up to isomorphism. The (left) localization of a ring \( A \) with respect to \( S \) exists if and only the (left) Orr condition is satisfied: for any \( s \in S \) and \( a \in A \), \( As \cap Sa = \emptyset \). The right localization is defined analogously. If there is both a left and right localization of a ring \( A \) with respect to \( S \), they are

isomorphic. We assume that the ring $B = S^{-1}A$ exists, and $M$ is some $A$-module. The localization $S^{-1}M$ is defined by a construction similar to that of $S^{-1}A$; $S^{-1}M$ is a $B$-module, and is canonically identifiable with the induced module $B \otimes M$.

The canonical homomorphism $\varphi: M \rightarrow S^{-1}M$, $m \mapsto 1 \otimes m$ is injective if and only if the module $M$ is $S$-torsion-free ($sm = 0$ implies $m = 0$). The mapping of the restriction of the $B$-submodules of $S^{-1}M$ to the $A$-submodules of $M$ that is defined by $N \mapsto \varphi^{-1}(N)$ is injective, and its image coincides with the set of submodules $L$ of $M$ such that $M/L$ is an $S$-torsion-free module. In the special case in which $M + A$ and $S^{-1}M = B$, the mapping $\varphi$ is an imbedding. It then follows that for any left ideal $N$ of the ring $B$, we have $\varphi^{-1}(N) = A \cap N$ and $S^{-1}(A \cap N) = N$. As a result, the mapping $N \mapsto A \cap N$ of left ideals of the ring $B$ into the left ideals of the ring $A$ is injective, and its image consists of all the left ideals $J$, for which the $A$-module $A/J$ is $S$-torsion-free.

Suppose that the localization of the ring $A$ with respect to $S$ exists; it then follows from the Orr condition that for any module $M$, the set $\text{tors}_S M = \{m \in M \mid sm = 0 \text{ for some } s \in S\}$ is an $A$-submodule in $M$. As a result, if the $A$-module $M$ is simple, it either has $S$-torsion ($M = \text{tors}_S M$), or it is $S$-torsion-free ($\text{tors}_S M = 0$).

Thus, the problem of describing simple $A$-modules splits into two parts: description of the $A$-modules that have $S$-torsion, and description of those that are $S$-torsion-free. In this section we describe the simple modules for the class of GWA under discussion that have $S$-torsion.

1.2. Imbedding of $A$ in a Euclidean ring. We recall the fundamental properties of a GWA $A [3]$. Since $D$ is a Noetherian domain, $A$ is a Noetherian domain and, by Goldy's theorem, it has a quotient field. The algebra $A = \bigoplus \{A_n \mid n \in \mathbb{Z}\}$ is $\mathbb{Z}$-graded $(A_n \subseteq A_{n+m} \forall n, m \in \mathbb{Z})$, where $A_n = Dv_n, v_n = X^n (n > 0), v_n = Y^n (n < 0), v_0 = 1$. Also, $\rho A$ and $A \rho$ are free $D$-modules with the natural system of generators $\{v_n \mid n \in \mathbb{Z}\}$. For any integers $i$ and $j$, we define the elements $(i, j)$ and $(i, j)$ of $D$:

$$v_i v_j = (i, j) = v_{i+j}(i, j).$$ (4)

It is clear that $(i, j) = \sigma^{i+j}(i, j)$. If $i > 0$ and $j > 0$, it is not difficult to calculate

$$i \geq j: (i, -j) = \Pi\{\sigma^k(a) | i-j+1 \leq k \leq i\}, (-i, j) = \Pi\{\sigma^k(a) | -i+1 \leq k \leq i+j\},$$

$$i \leq j: (i, -j) = \Pi\{\sigma^k(a) | 1 \leq k \leq i\}, (-i, j) = \Pi\{\sigma^k(a) | -i+1 \leq k \leq 0\}.$$ (5)

In the remaining cases $(i, j) = 1$.

Let $k$ be the quotient field of the ring $D$, i.e., $k = S^{-1}D$, where $S = D \setminus \{0\}$. It is clear that $S \subset A$ is a multiplicatively closed set that satisfies the left and right Orr condition in $A$. Let $B := S^{-1}A$ be the localization of $A$ with respect to $S$. Let $r$ be an automorphism of the field $k$, the ring of skew Laurent polynomials $k[X, X^{-1}; r]$ is the set of formal sums $\sum \{a_i X^i | i \in \mathbb{Z}\}$ with the natural addition (termwise) and multiplication $(\sum a_i X^i)(\sum a_j X^j) = \sum a_i a_j X^{i+j}$. The ring of skew Laurent polynomials is Euclidean (the left and right division axioms with remainder are satisfied) with respect to the "length" mapping $l: (\sum a_i X^m + \cdots + a_n X^n) = n - m, m < \cdots < n, a_n \neq 0, a_m \neq 0$. It is clear that $l(uv) = l(u) + l(v)$ for any $u$ and $v$. The ring of skew Laurent polynomials is isomorphic to the GWA $k(\sigma, \tau) = 1)$. The ring $k[X, X^{-1}; \tau]$ contains the subring $k[X; \sigma]$, which is called the ring of skew polynomials and is also a Euclidean ring with respect to the "degree" mapping deg: $\deg(\sigma_0 \cdots \sigma_n X^n) = n, a_n \neq 0$. It is clear that $\deg(uv) = \deg u + \deg v$ for any $u$ and $v$. Any Euclidean ring $E$ is a domain of principal ideals and right ideals. Recall that an $E$-module $M$ is simple if and only if $M = E/_{M}$ for some element $b \in E$ that is irreducible (in the usual sense: if $b = cd$, then either $c$ or $d$ is equal to unity); $E/_{M}$ is an isomorphism of $E$-modules if and only if both $b$ and $c$ are similar, i.e., there exists a $d \in E$ such that $I$ is the greatest common divisor of $c$ and $d$, and $bd$ is the least left multiple of $c$ and $d$.

LEMMA 1. The localization $B = S^{-1}A$ of a GWA $A$ of degree 1 with respect to a set $S = D \setminus \{0\}$ is a Euclidean ring that is isomorphic to the ring of Laurent polynomials $B \equiv k[X, X^{-1}; \sigma]$, where $k = S^{-1}D$ is the quotient field of the ring $D$.

1.3. Equivalence relation on Max $D$. Group $G = \langle \sigma \rangle$ acts naturally on the set of maximal ideals $M = \text{Max} D$ of a ring $D$. The set of orbits $\overline{M} = M/G$ splits into two disjoint classes, $\overline{M} = \text{Lin} \cup \text{Cyc}$, the linear and cyclic orbits, which respectively contain an infinite and a finite number of elements. An orbit is said to be degenerate if it contains at least one maximal ideal that contains a defining element of the algebra (otherwise, it is said to be nondegenerate). Then $\text{Lin} = \text{Lind} \cup \text{Cyc} = \text{Cycn} \cup \text{Cycl}$ is a partition of the sets $\text{Lin}$ and $\text{Cyc}$ into nondegenerate and degenerate orbits, respectively. Each linear orbit is canonically isomorphic to the set of integers $Z$, so it is naturally ordered; just as for $Z$, we can introduce