DECOMPOSABILITY OF MATRIX POLYNOMIALS WITH COEFFICIENTS POSSESSING QUASI-SIMPLE STRUCTURE

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The decomposability of matrix polynomials with commuting coefficients of quasi-simple structure into regular factors and the solvability of the corresponding matrix polynomial one-sided equations are investigated.

In [1] a theory was constructed that made it possible to solve the problem of separating a regular factor from a nonsingular matrix polynomial. However, for certain classes of matrix polynomials the general theory that has been developed is overly cumbersome and not very useful in practical applications. Moreover, in many special cases additional details and questions arise for which there are no answers. Therefore, it is necessary to establish decomposability conditions of matrix polynomials for certain classes on the basis of external sufficiently simple properties of these polynomials.

Suppose that \( \mathbb{C}_n \) is a ring of \((n \times n)\)-matrices over a field \( \mathbb{C} \). We wish to consider the polynomial
\[
A(x) = E + A_1 x + \cdots + A_s
\]
over \( \mathbb{C}_n \) and the corresponding polynomial matrix one-sided equations
\[
X + X^{-1} A_1 + \cdots + A_s = 0, \quad (2)
\]
\[
X + A_1 X^{-1} + \cdots + A_s = 0. \quad (3)
\]
We will assume that \( A_i A_j = A_j A_i \) and that all the coefficients possess a quasi-simple structure, that is, every characteristic root of the matrix \( A_i \), \( i = 1, \ldots, s \), with nonlinear elementary divisors is precisely of multiplicity two. In the present article conditions for the decomposability into factors of the matrix polynomial (1) and existence conditions for the complete set of solutions of Eqs. (2) and (3) are presented. The following lemma will be needed below.

**Lemma.** There exists a matrix \( S \in \text{GL}(n, \mathbb{C}) \) such that
\[
S A(x) S^{-1} = \bar{A}_1(x) \oplus \cdots \oplus \bar{A}_k(x) \oplus b_1(x) \oplus \cdots \oplus b_m(x),
\]
where \( \bar{A}_i(x) \), \( i = 1, \ldots, s \), is the upper triangular minor of order 2 with equal diagonal elements, \( b_j(x) \in \mathbb{C}[x], j = 1, \ldots, m, 0 \leq k \leq n/2, \) for the matrix polynomial (1).

**Proof.** If all the coefficients \( A_i \) are of simple structure, the lemma follows from Lemma 1 of Sec. 17 [2]. Therefore, let us assume that some coefficient \( A_t \) of the matrix polynomial \( A(x) \) has at least one quadratic elementary divisor. We now use induction on \( n \). The lemma is easily verified for \( n = 2 \). Let us assume that the inductive assumption holds for \( n - 1 \). There exists a matrix \( T \in \text{GL}(n, \mathbb{C}) \) such that
\[
T A_t T^{-1} = \begin{pmatrix}
J & 0 \\
0 & B
\end{pmatrix},
\]
where \( J \) is a Jordan cell of order 2. Since the spectra of the matrices \( J \) and \( B \) are disjoint,
Since the minor \( B(x) \), understood as a matrix polynomial, possesses pairwise commutative coefficients and since these coefficients are all of quasi-simple structure, then, by virtue of the induction assumption, it reduces, by means of similarity transformations, to quasi-diagonal form (4). The lemma is proved.

**THEOREM 1.** The degrees of the elementary divisors of the matrix polynomial \( A(x) \) is not greater than \( 2s \). The proof follows from the lemma and the fact that the system of elementary divisors of a quasi-diagonal matrix consists of combinations of the system of elementary divisors of the diagonal minors.

**THEOREM 2.** The matrix polynomial \( A(x) \) the degrees of whose elementary divisors are not greater than 3 decomposes into a product of unitary pairwise commuting factors the degrees of whose elementary divisors are not greater than 2.

**Proof.** It is sufficient to show that every diagonal minor of order 2 of the matrix (4) decomposes into a product of pairwise commuting factors of the form

\[
\begin{pmatrix}
\beta_{ii} \\
0
\end{pmatrix}
\begin{pmatrix}
x - \alpha_{ii}
\end{pmatrix}
\begin{pmatrix}
\beta_{ii} \\
0
\end{pmatrix}
\begin{pmatrix}
x - \alpha_{ii}
\end{pmatrix}
\]

For this purpose we use induction on \( s \). For \( s = 2 \), the theorem is valid, since none of the minors \( \tilde{A}_i(x) \) may possess an indecomposable matrix of the form

\[
\begin{pmatrix}
(x - \alpha)^2 & a(x) \\
0 & (x - \alpha)^2
\end{pmatrix}
\]

where \( a(\alpha) \neq 0 \) (such a matrix possesses an elementary divisor of degree 4). Let us consider the minor \( \tilde{A}_i(x) \) of degree \( s \) under the assumption that the theorem is valid for \( s - 1 \). Suppose that \( \alpha_{il} \) is a root of \( a_0^{(i)}(x) \). Its multiplicity, obviously, is not greater than 3. We denote by \( \nu_0 \) and \( \nu_1 \) the multiplicity of the root \( \alpha_{il} \) in the polynomials \( a_{0}^{(i)}(x) \) and \( a_{1}^{(i)}(x) \), respectively. The following cases are possible: 1) \( \nu_0 = 3, \nu_1 \geq 1 \); 2) \( \nu_0 = 2, \nu_1 \geq 1 \); 3) \( \nu_0 = 1, \nu_1 \geq 0 \). In each of these cases the following decomposition is found:

\[
\begin{pmatrix}
\beta_{ii} \\
0
\end{pmatrix}
\begin{pmatrix}
x - \alpha_{ii}
\end{pmatrix}
\begin{pmatrix}
\beta_{ii} \\
0
\end{pmatrix}
\begin{pmatrix}
x - \alpha_{ii}
\end{pmatrix}
\]

By the induction hypothesis we have found the required decomposition for the \((2 \times 2)\)-matrix \( \tilde{A}_i(x) \). The theorem is proved.

**THEOREM 3.** If the matrix polynomial \( A(x) \) possesses an elementary divisor of degree \( 2s \) mutually prime to its other elementary divisors, it is not decomposable into regular factors of lower degree.

**Proof.** If the conditions of the theorem are satisfied, there exists an indecomposable term in the direct sum (4) of the form

\[
\begin{pmatrix}
(x - \alpha)^s & a(x) \\
0 & (x - \alpha)^s
\end{pmatrix}
\]

where \( a(\alpha) \neq 0 \), moreover \( |\tilde{A}_j(\alpha)| = 0, b_j(\alpha) = 0, i \neq l, j = 1, \ldots, m \). By Theorem 3 of [3], the matrix (4), that is, the matrix \( A(x) \), is indecomposable.

A matrix polynomial is said to be regular if its leading coefficient is an invertible matrix.

**Definition 1.** A matrix polynomial that reduces, by means of similarity transformations, to quasi-diagonal form with regular minors along the leading diagonal is called a regular partition matrix.

**THEOREM 4.** A matrix polynomial \( A(x) \) the degrees \( k_i \) of whose elementary divisors satisfy the inequality \( 3 < k_i < 2s \) may be decomposed into a product of regular partition factors.

**Proof.** To prove the assertion it is sufficient to show that every minor of order 2 of the matrix (4) may be decomposed into regular factors. In fact, there does not exist among these minors indecomposable minors of the form...