The structure of continuous real-valued functions \( F(t) \) on the real line, such that for any fixed \( y \in \mathbb{R} \), the difference \( F(t + y) - F(t) \) is an almost periodic Bohr function, is investigated.

1. In this paper, continuous functions \( F(t): \mathbb{R} \rightarrow \mathbb{R} \) are considered for which for any fixed \( y \in \mathbb{R} \), the difference \( \psi_y(t) = F(t + y) - F(t) \) is almost periodic (a.p.) in the Bohr sense \( (\psi_y \in \mathbb{R}, \psi_y(t) \in \text{AP}(\mathbb{R})) \).

It is known that if \( \psi_y(t) \in \text{AP}(\mathbb{R}) \) \( \forall y \in \mathbb{R} \) and \( F(t) \) is bounded then \( F(t) \) is also an a.p. function (the Loomis theorem [1]). Bochner proved [2] that \( F(t) \in \text{AP}(\mathbb{R}) \) also if \( F(t) \) is bounded and uniformly continuous on \( \mathbb{R} \) and for some \( y_0 \neq 0, \psi_{y_0}(t) \in \text{AP}(\mathbb{R}) \).

Here we shall assume that in general \( F(t) \) is unbounded. We state the basic assertion.

**THEOREM 1.** If for any \( N > 0 \), the function \( \psi_y(t) \) is a.p. in the Bohr sense uniformly in \( |y| \leq N \), then the function \( F(t) \) is uniformly continuous and there exists a sequence \( \{a_n(t)\} \in \text{AP}(\mathbb{R}), n \in \mathbb{N} \) such that uniformly in \( t, a \in \mathbb{R} \)

\[
\lim_{n \to -\infty} \int a_n(t) \, du = F(t) - F(a).
\]

Moreover, the finite limit

\[
\lim_{t \to +\infty} \mathcal{T}^{-1}(F(T + a) - F(a)) = \langle F \rangle,
\]

exists uniformly in \( a \). Conversely, if for \( \{a_n(t)\} \in \text{AP}(\mathbb{R}) \), condition (1) is fulfilled uniformly in \( t \) for some \( a \) then the family \( \{\psi_y(t), y \in K\} \) is uniformly a.p. provided \( K \) is a compact set.

2. Functions \( \psi_y(t) \) are a.p. uniformly in \( y \in \mathbb{R} \) if and only if \( F(t) = mt + f(t), m \in \mathbb{R}, f(t) \in \text{AP}(\mathbb{R}) \).

The above stated theorem allows us to describe functions that are integral a.p. (N-integral a.p.) in the Wexler sense [3] and, in particular, to specify the form of numerical sequences \( \{t_n\}_{n=-\infty}^{\infty} \) such that the totality of sequences \( \{u_n\} = \{t_{n+1} - t_n\}_{n=-\infty}^{\infty}, \{i \in \mathbb{Z} \} \) is a.p. of equal degree (this is important in the theory of a.p. solutions of pulse systems [3, 4]).
2. Definition 1. A continuous function \( F(t) : \mathbb{R} \to \mathbb{R} \) is called \( \Delta \)-a.p. [\( N \)-\( \Delta \)-a.p.] if for each \( \varepsilon > 0 \) [and for each \( N > 0 \)] there exists an \( \omega > 0 \) [respectively, \( \omega \in \mathbb{R} \)] such that in any interval of length \( \ell \) and \( \omega \) can be found such that

\[
|F(t + \omega) - F(t) - (F(t' + \omega) - F(t'))| < \varepsilon \quad \forall t, t' \in \mathbb{R}
\]

[respectively, for \( t, t' \in \mathbb{R} \) such that \(|t - t'| < N\)].

Evidently, an \( \Delta \)-a.p. function is also an \( N \)-\( \Delta \)-a.p. function. It follows directly from the definition that if \( f(t) \in AP(R) \) the function \( mt + f(t) \) will be \( \Delta \)-a.p. The notions presented above generalize the \( [N] \)-integral a.p. functions introduced by Wexler.

Definition 2. A locally summable function \( f(t) : \mathbb{R} \to \mathbb{R} \) is integral a.p. [\( N \)-integral a.p.] if for any \( \varepsilon > 0 \) [and any \( N > 0 \)] there exists an \( \omega > 0 \) [respectively, \( \omega \in \mathbb{R} \)] such that in any interval of length \( \ell \) and \( \omega \) can be found such that

\[
|\int_{t}^{t+\omega} f(u) \, du - \int_{t'}^{t'+\omega} f(u) \, du| < \varepsilon \quad \forall t, t' \in \mathbb{R}
\]

[respectively, for \( t, t' \in \mathbb{R} \) such that \(|t - t'| < N\)].

Evidently, if \( f(t) \) is an integral a.p. [\( N \)-integral a.p.] function then for arbitrary \( a \in \mathbb{R} \) the function \( F(t) = \int_{a}^{t} f(u) \, du \in C(R) \) will be \( \Delta \)-a.p. [\( N \)-\( \Delta \)-a.p.]. Utilizing the identity

\[
\psi_{y}(t) - \psi_{y}(a) = (F(t + y) - F(t) - (F(t' + y) - F(t'))),
\]

we obtain that \( F(t) \) is a \( \Delta \)-a.p. function [an \( N \)-\( \Delta \)-a.p. function] if and only if the family of functions \( \psi_{y}(t) \) is a.p. uniformly in \( y \in \mathbb{R} \) [respectively, in \( y \) taking values in an arbitrary compact set \( K \subset \mathbb{R} \)].

We shall prove the uniform continuity of \( N \)-\( \Delta \)-a.p. functions. We fix \( N = 1 \) and an arbitrary \( \varepsilon > 0 \). Let \( \omega = \omega(\varepsilon, 1) > 0 \). Then in any interval of length \( \ell \) there exists an \( \omega \) such that for all \( t, t':|t - t'| < 1 \) the inequality

\[
|F(t + \omega) - F(t) - (F(t' + \omega) - F(t'))| < \varepsilon/2
\]

is fulfilled. On \([-\ell, 1]\), the function \( F(t) \) is uniformly continuous and given \( \varepsilon > 0 \) a positive \( \delta(\varepsilon) < 1 \) can be found such that for all \( t_1, t_2 \) with \(|t_1 - t_2| < \delta \) we have

\[
|F(t_1) - F(t_2)| < \varepsilon/2.
\]

Now let \( t^{(1)} < t^{(2)}, |t^{(1)} - t^{(2)}| < \delta \); on \([t^{(1)}, t^{(1)} + \ell]\) we specify an almost period \( \omega \) such that (4) is fulfilled. If we choose in (4) \( t = t^{(1)} - \omega \in [-\ell, 0]; t' = t^{(2)} - \omega \in [-\ell, 1]\), then in accordance with (4) and (5) \(|F(t + \omega) - F(t' + \omega)| = |F(t) - F(t')| < \varepsilon/2 \) and \(|F(t^{(1)}) - F(t^{(2)})| < \varepsilon \). Q.E.D.

It is now easy to prove the first part of the theorem. Indeed, let \( f_{n}(t) = 2^{n}(F(t + 2^{-n}) - F(t)) \). Then

\[
\int_{a}^{t} f_{n}(u) \, du = 2^{n} \int_{a}^{t} F(u + t) \, du - 2^{n} \int_{a}^{t} F(u + a) \, du = F(t) - F(a)
\]

for \( n \to \infty \) uniformly in \( t, a \in \mathbb{R} \) in view of the uniform continuity of \( F(t) \).

Let \( n_{0} \) be such that for all \( t, a \in \mathbb{R} \)

\[
\int_{a}^{t} f_{n_{0}}(u) \, du - (F(t + a) - F(a))| < 1.
\]

Subdividing both sides of (6) by \(|t|\) and noting that there exists, uniformly in \( a \), the finite limit \( \lim_{t \to a} F^{-1} \int_{a}^{t} f_{n}(u) \, du \), we verify that the limit (2) also exists uniformly in \( a \).

Conversely, let (1) be valid uniformly in \( t \) for some \( a \in \mathbb{R} \). Since \( f_{n}(t) \in AP(R) \) in accordance with [3, p. 297], a.p. functions \( f_{n}(t) \) are \( N \)-integral a.p. Then, however, \( F_{n}(t) = \int_{a}^{t} f_{n}(u) \, du \) are \( N \)-\( \Delta \)-a.p. functions. Taking into account that \( N \)-\( \Delta \)-a.p. functions form a linear space closed with respect to the limiting transition uniformly in \( t \in \mathbb{R} \), we verify the validity of the first part of the theorem.

As a corollary of the theorem, we note that for an arbitrary \( N \)-integral a.p. Wexler function \( f(t) \) there exists uniformly in \( a \) the finite mean