METHOD FOR STUDYING BOUNDARY VALUE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

A. M. Samoilenko and Le Lyiong Tai

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A constructive approach to the determination of an approximate solution of a boundary value problem with nonlinear boundary conditions \( g[z(0), z(T)] = 0 \) is proposed. Existence of the exact solution is proved, and error estimates for the constructed approximate solution are provided.

1. Let us consider a system of ordinary differential equations with strongly nonlinear boundary conditions

\[
\begin{align*}
\frac{dz}{dt} &= f(t, z), \\
g[z(0), z(T)] &= 0,
\end{align*}
\]

where \( f, g, z \) are points in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \).

Let us assume that the function \( f(t, z) \) is continuous in the domain

\[
(t, z) \in [0, T] \times \Omega,
\]

where \( \Omega \) is a closed bounded domain in \( \mathbb{R}^n \), and satisfies a boundedness condition, as well as Lipschitz condition with a matrix \( K \):

\[
|f(t, z)| \leq M, \quad M = (M_1, \ldots, M_n),
\]

\[
|f(t, z') - f(t, z'')| \leq K|z' - z''|,
\]

while the function \( g(u, v) \) is continuous in the domain

\[
(u, v) \in \Omega \times \Omega.
\]

Let \( A, B \) be some constant matrices, chosen so that \( \det B \neq 0 \). Instead of the boundary value problem (1), (2) we shall consider a boundary value problem of the form

\[
\begin{align*}
\frac{dz}{dt} &= f(t, z), \\
Az(0) + Bz(T) &= g[z(0), z(T)],
\end{align*}
\]
Let us introduce a change of variables
\[ t'(t) = x(t) + h(\xi, t), \]
where the function \( h(\xi, t) \), defined in the domain
\[ (t, \xi) \in [0, T] \times \Omega; \quad \Omega \subset E^n, \]
is continuously differentiable in \( t \) for \( t \in [0, T] \) and takes values in \( E^n \); \( x + h(\xi, \xi) \in \Omega, \xi \) is a parameter. Let \( D \) be the range of \( x \). Then under the change of variables (10) the boundary value problem (1), (2) becomes the boundary value problem
\[ \frac{dx}{dt} = f(t, x + h(\xi, t)) - \partial h(\xi, t)/\partial t, \]
\[ Ax(0) + Bx(T) = \varphi [x(0) + h(\xi, 0), x(T) + h(\xi, T)] - Ah(\xi, 0) - Bh(\xi, T) \]
in the domain
\[ (t, x) \in [0, T] \times D. \]

Let us apply now the numerical-analytical method to the boundary value problem (12), (13). For that, the following conditions have to hold:

1) \[ |f(t, x + h(\xi, t)) - \partial h(\xi, t)/\partial t| \leq M; \]

2) the set \( D_\beta \) of points \( x_0 \in E^n \), that together with their \( \beta \)-neighborhoods, where \( \beta = \frac{T}{2} M_1 + |(B^{-1} A + E) x_0| \), are contained in \( D \), is nonempty:
\[ D_\beta \neq \emptyset; \]

3) all the eigenvalues \( \lambda_j(Q) \) of the matrix \( Q = \frac{T}{\pi} K \) are contained in the unit circle:
\[ |\lambda_j(Q)| < 1, \quad j = 1, 2, \ldots, n. \]

Under the above assumptions the solution of the boundary value problem (12), (13) is obtained as the limit of the uniformly convergent sequence of functions defined by the recurrence relation
\[ x_{m+1}(t, x_0, \xi) = x_0 + \int_0^T \left[ f(s, x_m(s, \xi) + h(\xi, s), h(\xi, s)) - \partial h(\xi, t)/\partial t \right] ds - \frac{1}{T} \int_0^T (B^{-1} A + E) x_0, \]
while \( \xi \) is the solution of equation
\[ F(x_0, \xi) = \varphi (x_0 + h(\xi, 0), - B^{-1} A x_0 + h(\xi, T)) - Ah(\xi, 0) - Bh(\xi, T) = 0. \]

By [1], under the conditions 1-3 the sequence of functions (18) converges uniformly to a limit function \( x^* \), which is a solution of the boundary value problem
\[ \frac{dx}{dt} = f(t, x + h(\xi, t)) - \partial h(\xi, t)/\partial t - \Delta(x_0, \xi), \]
\[ Ax(0) + Bx(T) = 0, \]
\[ \Delta(x_0, \xi) = \frac{1}{T} \int_0^T \left[ f(s, x^*(s, x_0, \xi) + h(\xi, s), h(\xi, s)) - \partial h(\xi, t)/\partial t \right] dt - (B^{-1} A + E) x_0. \]

Let us show now that if \((x_0, \xi)\) is the solution of the system of algebraic equations
\[ F(x_0, \xi) = 0, \]
\[ \Delta(x_0, \xi) = 0, \]
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