A Variational Model of Preference Under Uncertainty

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Abstract

A familiar example devised by Daniel Ellsberg to highlight the effects of event ambiguity on preferences is transformed to separate aleatory uncertainty (chance) from epistemic uncertainty. The transformation leads to a lottery acts model whose states involve epistemic uncertainty; aleatory uncertainty enters into the state-dependent lotteries. The model proposes von Neumann-Morgenstern utility for lotteries, additive subjective probability for states, and the use of across-states standard deviation weighted by a coefficient of aversion to variability to account for departures from Anscombe-Aumann subjective expected utility. Properties of the model are investigated and a partial axiomatization is provided.

Key words: Ellsberg event ambiguity, decision under uncertainty, aleatory uncertainty, epistemic uncertainty, coefficient of aversion to variability

1. Introduction

This article argues that Daniel Ellsberg’s seminal criticism of subjective expected utility and the de Finetti-Ramsey-Savage conception of subjective probability (Ellsberg, 1961) has a lottery-acts interpretation that adheres to common intuitions about additive subjective probabilities while accounting for departures from subjective expected utility by a term for variability of utility across states. Although many models for decision making under uncertainty have been proposed to accommodate Ellsberg’s phenomenon of event ambiguity or vagueness,¹ the present perspective on his contribution seems not to have been developed elsewhere. However, I shall argue that it is supported by empirical results described in Heath and Tversky (1991) and other studies cited there, and that it provides a natural way to interpret Ellsberg’s phenomenon in certain circumstances.

Our discussion is framed against the expected utility background developed in Bernoulli (1738), Ramsey (1931), von Neumann and Morgenstern (1944), Savage (1954), and Anscombe and Aumann (1963). We follow the latter two works by supposing that the locus of a decision maker’s uncertainty is a set \( S \) of states of the world, exactly one of which is the true state. Its identity is not known to the decision maker, and his or her decision will not influence which state happens to be the true state.

For Savage (1954), the decision in conjunction with the true state fully determines the outcome or consequence experienced by the decision maker and others. Decision alternatives are acts \( f, g, \ldots \), each of which is a function from \( S \) into a set \( X \) of consequences.
If $f$ is taken and $s$ is the true state, then consequence $f(s)$ results. Savage’s preference relation $\succeq$ (is preferred or indifferent to) is defined on acts. His utility representation for $\succeq$ is

$$f \succeq g \iff \int_{s \in S} u(f(s))d\pi(s) \geq \int_{s \in S} u(g(s))d\pi(s),$$

where $\int u(f(s))d\pi(s)$ is the decision maker’s subjective expected utility for act $f$. In this model, $\pi$ is a unique finitely-additive probability measure defined on all subsets of $S$, and $u$ is a real-valued utility function on the set of consequences that is unique up to positive affine transformations ($u \rightarrow au + b$, $a$ and $b$ real numbers, $a > 0$). We refer to subsets of $S$ as events. The sure event $S$ has $\pi(S) = 1$, and $0 \leq \pi(A) \leq 1$ for every event $A$. Finite additivity says that $\pi(A \cup B) = \pi(A) + \pi(B)$ for every two disjoint events $A$ and $B$. Savage’s axioms (Savage, 1954; Fishburn, 1970) require $S$ to be infinite.

The version of the Anscombe-Aumann theory that I use here, following Fishburn (1970), has a finite state set $S = \{s_1, s_2, ..., s_n\}$. It applies $\succeq$ to the set $P$ of all functions $p, q, ...$ from $S$ into the set $\mathcal{P}$ of all finite-support probability distributions $p, q, ...$ on $X$. Members of $\mathcal{P}$ are lotteries, and elements in $P$ are lottery acts. We write $p = (p_1, p_2, ..., p_n)$ to denote that $p(s_i) = p_i$ for $i = 1, 2, ..., n$. The Anscombe-Aumann subjective expected utility model is

$$p \succeq q \iff \sum_{i=1}^{n} \pi(s_i)u(p_i) \geq \sum_{i=1}^{n} \pi(s_i)u(q_i),$$

where $\pi$ is a unique probability distribution on $S$ and $u$ is a linear functional on the lottery set $P$, i.e.,

$$u(\lambda p + (1 - \lambda)q) = \lambda u(p) + (1 - \lambda)u(q) \text{ for } 0 \leq \lambda \leq 1 \text{ and } p, q \in P.$$

As in Savage’s case, $u$ is unique up to positive affine transformations. A constant lottery act has $p(s_i) = p(s_j)$ for all $i$ and $j$. The specialization of the Anscombe-Aumann model to the set of constant lottery acts is tantamount to the von Neumann-Morgenstern utility model for preference between lotteries. The model described in the next paragraph reduces to the von Neumann-Morgenstern model when it is restricted to constant lottery acts.

For each lottery act $p$, let $u(p, \pi)$ and $\sigma_u(p, \pi)$ be the expected utility of $p$ and the standard deviation across states of $(u(p_1), ..., u(p_n))$ with respect to probability distribution $\pi$, respectively:

$$u(p, \pi) = \sum_{i=1}^{n} \pi(s_i)u(p_i),$$

$$\sigma_u(p, \pi) = \left\{\sum_{i=1}^{n} \pi(s_i)[u(p_i) - u(p, \pi)]^2\right\}^{1/2}.$$

The simplest representation for $(P, \succeq)$ that we consider in interpreting Ellsberg’s phenomenon is, for all $p, q \in P$,

$$p \succeq q \iff U(p) \geq U(q)$$