THEOREM 4. In addition to the hypotheses of Theorem 2 let (17) and (19) hold. Then if the series \[
\sum_{k=0}^{\infty} \sup_{k \leq l < k+1} g_n(x, t)
\]
converges uniformly with respect to \( n \geq 1, x \in E, \) and
\[
\sup_n \Delta \pi(n) \sum_{k=0}^{\infty} \{ \sup_{k \Delta \leq t < k \Delta + \Delta} (g_n(x, t) - \inf_{k \Delta \leq t < k \Delta + \Delta} g_n(x, t)) \} \Rightarrow 0,
\]
then
\[
\lim_{t \to c} U_n g_n(x, t) = e^{-\tau} \frac{1}{2} \int_0^\infty g(s) ds
\]
uniformly in \( x \in E. \)

LITERATURE CITED


OPERATOR CONDITIONS FOR \( C \)-DIFFERENTIABILITY

A. V. Bondar' and V. Yu. Romanenko

The concepts of dilation and rotation operators are introduced for maps of domains of infinite-dimensional Hilbert spaces. Sufficient conditions for \( C \)-differentiability along subspaces are established and theorems on operator conditions for \( \mathbb{R} \)-differentiable maps to be monogenic and holomorphic are proved.

In the present paper we continue the study of operator conditions for monogenicity and holomorphicity of maps of domains of Hilbert spaces started in [1], using the concepts and notation introduced there.

Let \( H \) be a complex Hilbert space, \( D \) be a domain in \( H, M \) be a subset of \( D, \) and \( f : D \to H \) be a map.

Definition 1. We shall say that the map \( f \) belongs to the family \( \mathcal{F}(D, M), \) if for any orthonormal frame \( \mathcal{E} \in \mathcal{O}(H) \) and any point \( a \in M \) one can find a frame of sequences \( \mathcal{E} \) at the point \( a \) with tangent frame \( \mathcal{E}, \) along which the derived operator \( L(f, \mathcal{E}, a) \) of the map \( f \) at the point \( a \) exists.

Let \( F \) be a closed \( C \)-linear subspace of \( H. \) We denote by \( \mathcal{O}(H, F) \) the set of all frames \( \mathcal{E} = \{e_a\}_{a \in U} \subset \mathcal{O}(H), \) for which \( \mathcal{E} = \mathcal{E}' \cup \mathcal{E}, \) \( \mathcal{U} = \mathcal{X} \cup \mathcal{X}, \) where \( \mathcal{E}' = \{e_a\}_{a \in \mathcal{X}} \) is an orthonormal basis in \( F, \) and \( \mathcal{E} = \{e_a\}_{a \in \mathcal{X}} \) is an orthonormal basis in the orthogonal complement \( F^\perp. \) Analogously, by

\[ \mathcal{H}(H, F) \] denotes the set of those subspaces \( E \in \mathcal{H}(H) \), for which there exists a real basis \( \xi \), belonging to \( \mathcal{O}(H, F) \). Clearly in this case \( E = E' \oplus E'' \), where \( E' \) and \( E'' \) are the closed real linear spans of the sets \( \xi' \) and \( \xi'' \) respectively.

Considering the subspace \( F \subset H \) fixed, we choose a subspace \( E_0 \in \mathcal{H}(H, F) \subset \mathcal{H}(H) \) and denote the conjugation operator corresponding to it \([1]\) by \( J \). For such a choice of \( E_0 \) the spaces \( F \) and \( F^\perp \) are invariant with respect to \( J \). Indeed if \( z \in F \) and \( z = z' + iz'' \), where \( z', z'' \in E_0 = E_0' \oplus E_0'' \), then \( z', z'' \in E_0 \) and hence \( Jz = z' - iz'' \in E_0' \oplus E_0'' = F \). One proves analogously that \( J[F^\perp] = F^\perp \).

Moreover, for any \( E \in \mathcal{H}(H) \) and in particular for any \( E \in \mathcal{H}(H, F) \) there is defined a unitary operator \( T_E : H \to H \) \([1]\) which acts on \( z = z' + iz'' \), where \( z', z'' \in E \), by the rule \( T_Ez = Jz' + iz'' \). \[ \text{Lemma 1.} \]

If \( x, y \in F \) and \( \| x \| = \| y \| = r \neq 0 \), then for any subspace \( E^* \in \mathcal{H}(F^+) \) one can find a subspace \( E^* \in \mathcal{H}(F) \) such that \( T_Ex = y \), where \( E = E' \oplus E'' \).

Proof. Without loss of generality one can assume that \( r = 1 \). We fix a subspace \( E^* \in \mathcal{H}(F) \) and consider two possible cases:

1. Let us assume that the vectors \( x \) and \( y \) are \( \mathbb{C} \)-linearly dependent, i.e., \( Jy = \mu x \), where \( \mu \in \mathbb{C} \). Let \( e_1 = \lambda x \), where \( \lambda \) is one of the roots of the number \( \mu \). We complete the one-element set \( \{e_1\} \) to an orthogonal basis \( \xi' = \{e_0 \} \), in \( F \) and we denote by \( E' \) the closed real linear span of the set \( \xi' \). Then \( E' \in \mathcal{H}(F) \), and \( E = E' \oplus E'' \in \mathcal{H}(H, F) \). Since \( |\mu| = |\lambda| = 1 \), one has

\[
T_Ex = T_E\lambda e_1 = \bar{\lambda} T_Ee_1 = \bar{\lambda} e_1 = J\lambda e_1 = J\lambda^2 e_1 = J\mu x = Jy = y.
\]

2. Now let the vectors \( x \) and \( y \) be \( \mathbb{C} \)-linearly independent. Then if \( \Theta = (x, Jy) \), then \( 1 + \text{Re} \Theta \neq 0 \) and hence the number \( v = \text{Im} \Theta (1 + \text{Re} \Theta)^{-1} \) is real and infinite. Let

\[
e_1 = (v + i)x + (v - i)Jy, e_1 = x + Jy.
\]

Since \( y \in F \), one has \( Jy \in F \), and hence it follows from the condition \( x \in F \) and (1) that the vectors \( e_1 \) and \( e_2 \) belong to \( F \). Moreover,

\[
(e_1, e_2) = (v + i)\| x \|^2 + (v - i)\| Jy \|^2 + (v + i)(x, Jy) + (v - i)\| Jy \|^2 = 2(1 + \text{Re} \Theta)\nu - \text{Im} \Theta = 0.
\]

We complete the set \( \{e_1, e_2\} \) to an orthogonal basis \( \xi' = \{e_0 \} \), of \( F \) and denote by \( E' \) the closed real linear span of the set \( \xi' \). Considering the equality

\[
x = \frac{1}{2i} [e_1 - (v - i)e_2],
\]

which follows from (1), letting \( E = E' \oplus E'' \), we have

\[
T_Ex = \frac{1}{2i} (T_Ee_1 - (v - i)T_Ee_2) = \frac{1}{2i} (e_1 - (v - i)e_2) = \frac{1}{2i} [(v - i)Jx + (v + i)y - (v - i)(Jx + y)] = y.
\]

Lemma 1 is proved.

If \( a \) is a point of the domain \( D \subset H \) and \( f : D \to H \) is a map belonging to \( \mathcal{S} (D, \{a\}) \), then the set of derived operators \( \mathcal{P}(f, a) \) of the map \( f \) at the point \( a \) is not empty. For any operator \( L (f, \xi, a) \in \mathcal{P} (f, a) \) we write its left and right polar representations \([2, p. 354]\)

\[
L(f, \xi, a) = R^{(l)}_{\xi} U_{\xi}, L(f, \xi, a) = U_{\xi} R^{(r)}_{\xi}.
\]