In the case \( n = 2k \) the set \( H_2 \) is empty, and \( H_3 \) contains \( n! - \alpha(n + 1) \) permutations and then the number of \((n + 1)\)-periodic digraphs is equal to \( 1 + 1/2 |H_3| \). This proves the theorem.

LITERATURE CITED


A PROBLEM OF APPROXIMATE INTEGRATION, ARISING
IN THE THEORY OF QUEUEING SYSTEMS

S. V. Pereverzev and Zh. E. Myrzanov UDC 517.5

1. Formulation of the Problem. We denote by \( L_2(Q) \) the space of functions \( g(x, y) \), square integrable on \( Q = [-1,1] \times [-1,1] \), with the norm

\[
\| g \| = \left( \int_{-1}^{1} \int_{-1}^{1} g^2(x, y) \, dx \, dy \right)^{1/2},
\]

while \( L^{r,s}_2(Q) \), \( r = 1, 2, \ldots, s = 1, 2, \ldots, \) is the space of functions \( g(x, y) \) for which the partial derivatives \( g^{(r,s)}(x, y) \) belong to \( L_2(Q) \). The analogous spaces of functions of one variable, defined on \([-1,1]\), will be denoted by \( L_2 \) and \( L^{r}_2 \), respectively.

Assume that the self-adjoint operator

\[
Hf(x) = \int_{-1}^{1} g(x, y) f(y) \, dy
\]

with kernel \( g(x, y) \in L^{r}_2 \) has unity as a simple eigenvalue.

In this paper we shall consider the problem of the restoration of the integral

\[
(F, u) = \int_{-1}^{1} F(x) u(x) \, dx
\]

from the information

\[
Hx^i \equiv \int_{-1}^{1} g(x, y) y^i \, dy = \mu_i(x), \quad i = 0, N,
\]

\[
HF(x) \equiv \int_{-1}^{1} g(x, y) F(y) \, dy = \mu_{N+1}(x),
\]

where \( F(x) \in L_2 \) is some fixed function, \( u(x) \) is the eigenfunction of the operator \( H \), corresponding to the eigenvalue \( \lambda = 1 \), i.e., \( u(x) = \int_{-1}^{1} g(x, y) u(y) \, dy \) \quad \text{ and } \quad \int_{-1}^{1} u(x) \, dx = 1 \).

The problem of the restoration of the functional \((F, u)\) from information of the form (1), (2) arises frequently in queueing theory [1] in the following situation. Let \( u(x) \) be the probability density of the stationary distribution of a Markov chain and let \( g(x, y) \) be the conditional density of the transition from \( y \) to \( x \). Then \( u(x) \) is an eigenfunction of an integral operator with kernel \( g(x, y) \), corresponding to the eigenvalue \( \lambda = 1 \) and normalized in the manner indicated above. Generally, the transition density is not known exactly.
and one has only information of the form (1), (2). It presents interest to restore the value of a definite linear functional (for example, the average waiting time of a customer in a queuing system) of the performance index \( u(x) \) of some process.

We mention that the problem of the restoration of the functional \((F, u)\) from the information (1), (2) has been pointed out to the authors by I. N. Kovalenko.

2. Auxiliary Statements. Let \( S_N \) be the operator which associates to each function \( f(x) \in L_2 \) the partial sum of order \( N \) of its Fourier-Legendre series [2], and let \( S_{N,x} \) and \( S_{N,y} \) be the operators defined by the relations

\[
S_{N,x}g(x, y) = \sum_{i=0}^{N} a_i(y) P_i(x), \quad a_i(y) = \frac{1}{i!} \int g(x, y) P_i(x) \, dx,
\]

\[
S_{N,y}g(x, y) = \sum_{j=0}^{N} b_j(x) P_j(y), \quad b_j(x) = \frac{1}{j!} \int g(x, y) P_j(y) \, dy,
\]

where \( P_k(\cdot), k = 0, N, \) are the Legendre polynomials of order \( k \), orthonormalized on the segment \([-1, 1]\).

From Lebesgue's inequality (see, for example, [2, p. 34]), the boundedness for each \( N \) of the norms \( \|S_N\|_{L_2} \leq L \), and from known estimates of the error of the best approximation of functions from \( L_2^r \) by algebraic polynomials of degree at most \( N \) there follows that for any function \( f(x) \in L_2^r \) we have the inequality

\[
\|f - S_Nf\| \leq \alpha_r N^{-r} \|f^{(r)}\|, \quad (3)
\]

where \( \alpha_r \) is a constant, independent of \( f \) and \( N \).

We mention that if the information (1) is given, then the functions \( S_{N,x}g(x, y) \) and \( S_{N,y}g(x, y) \) are linear combinations of the functions \( \{x^i\}_{i=0}^{N} \) and \( \{y^j\}_{j=0}^{N} \), respectively, with certain definite coefficients.

We denote by \( H_N(H) \) the operator defined by the equality

\[
H_N(H)f(x) = \int \left( S_{N,x}g(x, y) + S_{N,y}g(x, y) - S_{N,x} S_{N,y}g(x, y) \right) f(y) \, dy.
\]

**Lemma.** If \( g(x, y) \in L_2^r, s(Q) \), then we have the inequalities

\[
\| H - H_N(H) \| \leq \alpha_{rs} N^{-r-s} \|g^{(r,s)}\| = \delta_N, \quad (4)
\]

\[
\| H(H - H_N(H))\| \leq \alpha_{rs}^2 N^{-2r-2s} \|g^{(r,s)}\| \|g^{(r,s)}\| \|g^{(r,s)}\|, \quad (5)
\]

where \( \alpha_{rs} \) is some constant, not depending on \( g(x, y) \) and \( N \).

**Proof.** Making use of inequality (3) and Fubini's theorem, we find

\[
\| H - H_N(H) \| \leq \int \left\{ \int \left| g(x, y) - S_{N,x}g(x, y) - S_{N,y}g(x, y) \right| \, dy \right\} \, dx = \int \left\{ \int \left| S_{N,x}g(x, y) - S_{N,y}g(x, y) \right| \, dy \right\} \, dx
\]

\[
= \alpha_{rs}^2 N^{-r} \left\{ \int \left[ g^{(r,s)}(x, y) - S_{N,y}g^{(r,s)}(x, y) \right] \, dy \right\} \, dx \leq \alpha_{rs}^2 N^{-2r-2s} \|g^{(r,s)}\| \|g^{(r,s)}\|, \quad (6)
\]

where \( \alpha_{rs} = \alpha_r \alpha_s \).

Now we prove inequality (5). We note that the function \( \phi(x) = (H - H_N(H))Hf(x) \) is orthogonal to polynomials of degree at most \( N \) and, therefore, \( H\phi(x) = (H - HS_N)\phi(x) \). From here, making use of the Cauchy-Bunyakovsky inequality and (3), we obtain

\[
\| (H - HS_N) \varphi (\cdot) \| \leq \left( \int \left\| g(x, \cdot) - S_{N,y}g(x, \cdot) \right\|^2 \, dx \right)^{1/2} \leq \alpha_r N^{-r} \int \left\| g^{(r,s)}(x, \cdot) \right\|^2 \, dx \right)^{1/2} \| \varphi \| = \alpha_r N^{-r} \|g^{(r,s)}\| \| \varphi \|.
\]