ASYMPTOTIC CONDITIONS AND INFRARED DIVERGENCES IN QUANTUM ELECTRODYNAMICS

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A definition which is free of infrared divergences is proposed for the S matrix of a relativistic theory of interacting charged particles. This is achieved by a modification of the asymptotic condition and the introduction of a new space of asymptotic states. This state differs from the Fok space, but is separable and relativistically and gauge invariant. The mass operator has no nonvanishing discrete eigenvalues.

In the present paper, we shall discuss some aspects of the scattering problem in the relativistic quantum theory of interacting charged particles and photons. The main result is the description of a space of asymptotic states for such a system and the definition of an S matrix that is free of infrared divergences.*

The infrared catastrophe has frequently been discussed, the first occasion being the classical paper of Bloch and Nordsieck in 1937 [1]. The physical reasons for infrared divergences are well understood and they do not lead to any physical problems. However, the generally accepted formal treatment of the infrared catastrophe is not, in our view, completely satisfactory.

In textbooks on quantum electrodynamics, the reader must wrestle with infrared divergences and sum the probabilities of a transition from a given initial state to all final states, which include not only detectable particles but also an arbitrary number of "soft" photons (see [2]). An important role in the justification of this approach is played by the asymptotic formulas for the scattering amplitudes in the case when the artificially introduced photon mass tends to zero. The general form of these formulas was derived in the papers of Yennie et al. [3].

In the classical method just described, the cross sections and not the matrix elements are regarded as the primary objects. The initial and final states are treated as asymmetric and an S matrix is not defined at all. One is naturally led to ask whether these features are unavoidable and due to the physical nature of the problem or whether there exists an alternative approach to infrared singularities in which an S matrix can be defined. In the present paper, we attack the problem in this manner and propose a version of the asymptotic condition which is specially suited to a relativistic system of charged particles and makes possible a correct definition of an S matrix.

Our point of departure is Chung's important paper [4]. Chung surmised how one can choose states containing a charged particle and a superposition of an infinite number of photons in such a way that the matrix elements of the Feynman–Dyson S matrix between these states are finite and nonzero. Chung's generalization of the construction of these states for the case of several charged particles is too unsophisticated; in particular, it ignores the infinite Coulomb phase.

Kibble [5] made some important advances on Chung's work. He introduced a very large space of asymptotic states and showed that the Feynman–Dyson S matrix can be correctly defined in this state as a unitary operator. Kibble's space is nonseparable and contains states with an infinite number of soft photons. One can distinguish separable subspaces of Kibble's space which are mapped into one another by the S

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matrix. However, there is no stable separable subspace that is mapped into itself. This is connected with the infinite Coulomb phase contained in the S matrix. Kibble's analytic apparatus is based on the asymptotic formulas for the many-particle Green's function near the mass shell for the charged particles.* Thus, Kibble's definitions are based on the complete solution of the dynamical problem and are therefore very cumbersome.

Our approach differs from these approaches in what we modify not only the space of asymptotic states but also the very definition of the scattering operator. This enables us to compensate the Coulomb phase automatically and our space of asymptotic states is separable and is no richer in states than the Fok space. The complete procedure is suggested by the nonrelativistic theory of scattering by a long-range potential and has a simple physical interpretation. We are not forced to solve the complete equations of quantum electrodynamics in order to implement our program. Thus, we derive Chung's formulas without laborious calculations and obtain their correct generalization in the case of an arbitrary number of charged particles and photons in the initial and final states.

From the methodological point of view, the main result of our paper is a relativistically and gauge invariant definition of the S matrix and the space of asymptotic states of the charged particles.

In the present paper, we take the example of Coulomb scattering to explain the main idea of our approach. The nub of the idea is that in the definition of the wave operators we do not take $\exp\{-iH_0t\}$ but a more suitable operator $U_{as}(t)$ as the operator of the asymptotic dynamics. The choice of this operator is based on a natural physical condition, namely, the wave packets $U_{as}(t)\Psi$ at large $|t|$ must correspond to the classical motion of widely separated charged particles. The actual choice of $U_{as}(t)$ for quantum electrodynamics is discussed in Sections 2 and 3. In the next section, we introduce and discuss a space of states, different from to Fok's space, for charged particles and photons. In Section 5, we explain why this space can be used naturally as the space of asymptotic states and we give the final definition of the S matrix and compare our results with those of Chung.

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1. Nonrelativistic Coulomb Scattering

The scattering of a nonrelativistic particle by a Coulomb potential may serve to illustrate the main idea of our approach. The Hamiltonian of the system has the form

$$H = \frac{\mathbf{p}^2}{2m} + \frac{g}{r} = H_0 + V,$$

where m is the mass of the particles and g is the product of the charges of the particle and the scattering center. The asymptotic behavior of the potential $V(t)$ in the interaction representation can be easily calculated. One must note that

$$\mathbf{r}(t) = \frac{\mathbf{p}}{m}t + \mathbf{r},\quad \mathbf{p}(t) = \mathbf{p},$$

i.e., as $|t| \to \infty$

$$V(t) = \frac{mg}{p|t|} + O\left(\frac{1}{t^2}\right).$$

The first term of this asymptotic expression cannot be integrated with respect to the time in the neighborhood of infinity and its contribution to the dynamics cannot be neglected, even for $|t| \to \infty$. In other words the asymptotic dynamics is not described by H_0 but by the explicitly time-dependent operator †

$$H_{as}(t) = H_0 + V_{as}(t) = H_0 + \frac{mg}{p|t|}.$$

The wave packets $\psi(\mathbf{r}, t)$ satisfy the asymptotic Schrödinger equation

$$i\frac{d}{dt}\psi(\mathbf{r}, t) = H_{as}(t)\psi(\mathbf{r}, t), \tag{1}$$

* Such formulas have a long history. A generating functional for the Green's functions which takes into account this asymptotic behavior rigorously was obtained by Fradkin [6].

† Here, we have used the fact that the expression for $V_{as}(t)$ is the same in both the Schrödinger and the interaction representation.