NECESSARY AND SUFFICIENT CONDITION OF EXTREMUM FOR THE

LEBESGUE–STIELTJES INTEGRAL ON A CLASS OF DISTRIBUTIONS

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We consider the optimization problem for the Lebesgue–Stieltjes integral defined on the set of distribution functions with two fixed power moments. The Krein–Nudel’man theorem on the necessary and sufficient condition of extremum of this functional is generalized to the case of a piecewise-continuous integrand.

Let $I(F)$ be the functional $I(F) = \int_0^Q g(x) \, dF(x)$ defined on the set $K$ of distribution functions $F$ with two fixed power moments. In this paper, we generalize the theorem of Krein and Nudel’man [1] on the necessary and sufficient condition of infimum (supremum) of the functional $I(F)$, $F \in K$, to the case of a function $g(x)$ with discontinuities of the first kind. This theorem is useful in reliability theory for optimization of functionals that characterize the operating performance of complex systems.

STATEMENT OF THE PROBLEM AND THE MAIN RESULT

Given is a piecewise-continuous function $g(x)$ with finitely many discontinuity points. The function is bounded (both from above and from below) and is defined at each point of the interval $[0, Q]$, $0 < Q < \infty$.

Given are the real quantities $s_1$ and $s_2$ satisfying the inequalities $0 < s_1 < Q$, $s_1^2 < s_2 < s_1 Q$. Denote by $K$ the set of distribution functions (d.f.) $F$ such that

\[
F(0^-) = 0, \quad F(Q + 0) = 1, \quad \int_0^Q x \, dF(x) = s_1,
\]

\[
\int_0^Q x^2 \, dF(x) = s_2.
\]

Under the above restrictions on the function $g(x)$ and the moments $s_1$ and $s_2$, the functional

\[
I(F) = \int_0^Q g(x) \, dF(x), \quad F \in K,
\]

exists (in the sense of the Lebesgue–Stieltjes integral) with its exact lower and upper bounds. Since the function $g(x)$ is discontinuous, these bounds are not necessarily attained.

If the function $g(x)$ is continuous, then we have Theorem 1 [1].

Denote by $E$ the set of extreme distribution functions of the set $K$ (see below about the set $E$).

THEOREM 1. The infimum (supremum) of the functional $I(F)$, $F \in K$, is attained on the d.f. $F_0 \in E$ with points of increase $x_i$, $i = 1, 2$ or $i = 1, 2, 3$ if and only if there exists a polynomial of not higher than second degree with the following properties;

1) $U(x; F_0) = g(x)$, $\bar{U}(x; F_0) = g(x)$,

2) $U(x; F_0) \leq g(x)$, $\bar{U}(x; F_0) \geq g(x)$ for all $x$ from $[0, Q]$.

*In [1] Theorem 1 is proved for an arbitrary fixed number of generalized moments and a continuous function $g(x)$. Here this theorem is stated in application to our specific problem.

The main result of our study is Theorem 2. It generalizes Theorem 1 to the case of a discontinuous function \(g(x)\) from the class introduced above.

Let \(g_*(x_i) = \min\{g(x_i - 0), g(x_i + 0)\}\), \(g^*(x_i) = \max\{g(x_i - 0), g(x_i + 0)\}\).

**Definition 1.** We say that the infimum (supremum) of the functional \(I(F)\), \(F \in K\), is evaluated on the d.f. \(F_0 \in E\) if

\[
\inf_{F \in K} I(F) = \sum_{i} g_*(x_i) p_i(\bar{x}) \quad \sup_{F \in K} I(F) = \sum_{i} g^*(x_i) p_i(\bar{x}),
\]

where \(x_i\) is a point of increase of the d.f. \(F_0\), \(p_i(\bar{x})\) are the jumps of the d.f. at these points, \(\bar{x}\) is the vector of points of increase.

**THEOREM 2.** The infimum (supremum) of the functional \(I(F)\), \(F \in K\), is evaluated on the d.f. \(F_0 \in E\) with points of increase \(x_i, i = 1, 2\) or \(i = 1, 2, 3\) if and only if there exists a polynomial of not higher than second degree with the following properties:

1) \(U_0(x; F_0) = g_*(x_0)\) \((U_0(x; F_0) = g^*(x_0))\),

2) \(U_0(x; F_0) \leq g(x)\) \((U_0(x; F_0) \geq g(x))\) \(\forall x \in [0; Q]\).

If \(g(x)\) is a continuous function, then \(g_*(x_i) = g^*(x_i) = g(x_i)\) and the infimum (supremum) of the functional \(I(F)\) is attained, i.e., in this case Theorem 2 coincides with Theorem 1.

The proof of Theorem 2 relies on the properties of the set \(E\) of extreme distribution functions and the set \(U\) of associated polynomials.

**EXTREME DISTRIBUTION FUNCTIONS AND THE CORRESPONDING POLYNOMIALS**

The d.f. \(F(x)\) from \(K\) is called extreme if it cannot be represented in the form \(F(x) = \frac{1}{2}F_1(x) + \frac{1}{2}F_2(x)\), where \(F_1(x)\) and \(F_2(x)\) are different d.f.s from \(K\). We know [2] that \(\inf_{F \in K} I(F) = \inf_{F \in E} I(F)\), where \(E\) is the set of extreme points (d.f.s) in \(K\) which are two- or three-step d.f.s. These distribution functions are uniquely determined by their points of increase from moment conditions.

Identify in \(E\) the subset \(E'\) that consists only of three-step d.f.s. Let \(x_1, x_2, x_3\) \((x_1 < x_2 < x_3)\) be the points of increase of the d.f. \(F\) from \(E'\), and \(\bar{x} = (x_1, x_2, x_3)\) the vector of these points. The jumps at the points of increase are determined from (1) using the formulas

\[
p_i(\bar{x}) = \frac{Q_i(x)}{Q_i(x_i)}, \quad i = 1, 3,
\]

where \(Q_i(x) = Q(x)/(x - x_i), \quad Q(x) = (x - x_1)(x - x_2)(x - x_3)\).

The inequalities \(p_i(\bar{x}) > 0, i = 1, 2, 3\), that hold for all d.f.s from \(E'\) are equivalent to the inequalities

\[
0 \leq x_1 < B(x_2) < x_2 < B(x_3) < x_3 \leq Q,
\]

where the function \(B(x)\) is given by

\[
B(x) = \frac{s_3 - s_1 x}{s_1 - x}, \quad x \in \{0; B(Q)\} \cup [B(0); Q]\).
\]

From the properties of the function \(B(x)\) it follows that

\[
0 \leq B(x_0) \leq B(Q) < s_1 < B(0) \leq B(x_3) \leq Q.
\]

Inequalities (5) and (6) show that the first and the third points of increase of all d.f.s from \(E'\) are separated by a finite interval, whereas the first and the second or the second and the third points of increase may be infinitely close with a common limit \(B(x_3)\) or \(B(x_1)\), respectively.

In the set \(E'\) define the metric \(\rho_1(F_1, F_2) = \max_{1 \leq i < 3} |x_i - y_i|\), where \(F_1, F_2 \in E', x_i\) and \(y_i\) are the points of increase of \(F_1\) and \(F_2\), respectively. The distance between two two-step d.f.s from \(E\) is also measured in this metric. In order to measure the distance between a three-step d.f. \(F_1\) with points of increase \(x_1, x_2, x_3\) and a two-step d.f. \(F_2\) with points of increase \(y_1\) and \(y_2\), we represent the vector of points of increase of the d.f. \(F_2\) in the form of one of the triples \((x_3, y_1, y_2), (y_1, x_3, y_2),\) or \((y_1, y_2, x_3)\), where the point \(x_3\) satisfies the appropriate inequalities (in the first triple, \(0 \leq y_3 \leq y_1\), in the second triple \(y_1 < y_3 \leq y_2\), and in