The paper considers functions defined by R-transducers on the set of binary representations of real numbers, in particular addition and multiplication operations, and examines the properties of the arithmetic of these R-numbers.

Addition and multiplication are the simplest and most basic operations on real numbers. One of the formalizations of real numbers in classical analysis is by binary representation, which in the modern context corresponds to machine representation of numbers in computers. This formalization leads to a definition of R-number [1] as an arbitrary element of the set $D = ((0 \cup 1 \{0, 1\}^*) \setminus \{0\} \cup 1 \{0, 1\}^*) \setminus \{0, 1\}^\infty$, where the symbols $\setminus$ and $1$ are the "binary" point and "minus one". To each R-number $x$ obviously corresponds a real number $\|x\|$ [1]. In the class of functions $f: D \to D$ we distinguish the subclass of R-functions (defined by R-transducers), the class of C-computable functions, and the class of finite R-functions [1].

The last two classes are the real functions defined by Turing machines and by finite R-transducers. Real functions are defined in [2] by finite Mealy automata and generalized sequential machines. Each continuous real function $g$ is representable as an R-function $f$ so that $g = f [1]$. This applies, in particular, to addition and multiplication functions of real numbers. In this paper, we provide rigorous definitions of R-functions of addition and multiplication, examine the resulting R-arithmetic, and compare the laws of R-arithmetic with the laws of machine arithmetic [3].

The addition R-function may be defined by a finite R-transducer. However, no finite R-transducer exists defining the multiplication function. Some R-functions are not defined by finite R-transducers because finite R-functions take rational values at rational points (Lemma 3). The class of finite R-functions contains continuous nowhere differentiable functions. All R-functions are Riemann integrable, but there exist R-functions $\varphi$ and $h$ such that the function $f(x) = \varphi(h(x))$ is no longer Riemann integrable. Moreover, continuous real functions defined by finite R-transducers are described by special recursive systems of equations. Computable functions of a natural argument are known to be definable by recursive systems of Herbrand–Gödel equations. The Herbrand–Gödel approach is extended in this paper so that computable real functions are defined by special recursive systems of equations. In conclusion we show that continuous operators in a Hilbert space correspond to certain manipulation techniques of infinite arrays associated with R-transducers, while continuous real functions are also representable by infinite labeled trees.

We define two addition R-functions $g_1^0$, $g_1$ and multiplication R-function $g_2$:

$$\begin{align*}
g_1^0(x, y) &= \tilde{x} + \tilde{y}, \quad g_1^0(x, y) \in \mathbb{D}_0 \cup K_0 \cup \{0\}; \\
g_2(x, y) &= \tilde{x} \tilde{y}, \quad g_2(x, y) \in \mathbb{F}_0, \text{ if } (x, y) \in \mathbb{F}_0 \text{ or } \tilde{x} \tilde{y} = 0, \text{ otherwise } g_2(x, y) \in \mathbb{D}\setminus\mathbb{F}_0; \\
g_1(x, y) &= \tilde{x} + \tilde{y}, \quad g_1(x, y) \in \mathbb{F}_0 \text{ if } \tilde{x} + \tilde{y} = 0 \text{ or } \exists \alpha \in \{1, -1\}, i \in \mathbb{N}^\ast((x, y) \subset \alpha \sum_{\alpha} \bigvee_{\alpha} (x, y) \subset 0 \setminus \{0\} \text{ and } (x, y) \in \mathbb{F}_0, \text{ otherwise } g_1(x, y) \in \mathbb{D}\setminus\mathbb{F}_0. \end{align*}$$

The pair of R-functions $g_1^0$ and $g_2$ should have the usual properties of addition and multiplication (commutativity, associativity, distributivity of $g_2$ by $g_1^0$). From the computational viewpoint, it is more natural to consider the R-function $g_1$ (and not $g_1^0$), because it is defined by an R-transducer representing the addition algorithm that constructs the result $g_1(x, y)$ from the high-order to the low-order bits and allows correction (when the current bit overflows).
LEMMA 1. The operation \( g_1 \) satisfies the commutativity property but does not satisfy the properties of associativity and distributivity of \( g_2 \) by \( g_1 \).

Proof. The identity \( g_1(x, y) = g_1(y, x) \) is obvious. It remains to note the following counterexamples: \( g_1(g_1(x, y), z) \neq g_1(x, g_1(y, z)), g_2(g_1(x, y), v) \neq g_1(g_2(x, v), g_2(z, v)) \) for \( x = y = 0 \) and \( z = 0 \), which contains the law of commutativity but does not contain the associative and distributive laws.

A similar situation arises in machine arithmetic [3, pp. 53-54], which contains the law of commutativity but does not contain the associative and distributive laws.

The R-function \( g_1^0 \) is the real addition function with value set \( G \cup K \cup \{0\} \). In general, any real function \( \tilde{f} \) represented by the R-function \( f \) may be represented by the R-function \( g \) with value set \( G \cup K \cup \{0\} \). This follows from the next lemma.

LEMMA 2. For any R-transducer \( A \) there exists an R-transducer \( B \) such that \( \text{Dom}(f_A) = \text{Dom}(f_B) \) and \( \| f_A(x) \| = \| f_B(x) \|, f_B(x) \in G \cup K \cup \{0\} \) for any \( x \in \text{Dom}(f_A) \).

The proof is by constructing an R-transducer \( B \) that "computes the output value with a deficiency". Each R-transducer \( A = (K, H, q_0) \) processes superwords in the alphabet \( \Sigma_0 \) into words or superwords in the same alphabet [1]. Without extending the functional capabilities, we convert the R-transducer commands into macros by writing them as productions of the form \( q_0 \rightarrow \beta p \), where \( q, p \in K, \alpha, \beta \in \Sigma_0^* \). This production implies that the transducer \( A \) in state \( q \) reads \( \alpha \) from the input, delivers \( \beta \) to the output, and goes to state \( p \). Such a generalized R-transducer will be called an OR-transducer. It has a finite or countable set of productions \( H \). It is required that none of the words in the set \( \{u | v \in H\} \) is a left subword of another word. Otherwise \( A = (K, H, q_0) \) is a non-deterministic OR-transducer. Like the R-transducer [1], the OR-transducer \( A \) defines a partial R-function \( f_A: D \rightarrow D \). If \( \text{Dom}(f_A) = D \), then \( f_A \) is a R-function.

The function \( f: D \rightarrow D \) is called metareal iff \( f(x) = f_1(f_2(...(f_n(x))...)) \) for some \( n \in \mathbb{N} \) and R-functions \( f_1, \ldots, f_n \). Let \( f_1(x) = \phi(x) \) and show that metareal functions in general are not Riemann-integrable, whereas R-functions are Riemann-integrable.

THEOREM 1. There exists a metareal function \( f(x) = \phi(h(x)) \) such that the real function \( \tilde{f} \) is not Riemann-integrable on \([0, 1]\).

Proof. Define the R-function \( h \) so that for any \( v \in \{0, 1\}^\infty \), the value \( h(0^\infty v) \) equals \( f_A(0^\infty v) \), where \( f_A \) is the R-function defined by the OR-transducer \( A \) with the macro productions:

\[
\begin{align*}
q_0 & \rightarrow 0^\infty, \quad q_1 00 \rightarrow 12g, \quad ga \rightarrow 1g \quad (a = 0, 1), \\
qa0 & \rightarrow 11q_0 (ab \in \{0, 1\}^2 \setminus \{00\}), \quad qa000 \rightarrow 112g, \\
qa0 & \rightarrow 1111qa (abcd \in \{0, 1\}^4 \setminus \{0000\}), \\
qa0 & \rightarrow 112q_{k+1} (a_1 \ldots a_k \in \{0, 1\}^k \setminus \{0^k\}), \\
qa0 & \rightarrow 12g \quad (k = 3, 4, \ldots).
\end{align*}
\]

Let \( f(x) = \phi(h(x)) \), \( x \in D \). The total sum of lengths of the intervals on which \( f(x) = 1 \) for \( 0 \leq x < 1 \) is \( 1/4 + 3/4(1/2^2 + (4^3 - 1)/4^3)(1/4^4 + (4^3 - 1)/4^3)(1/4^5 + \ldots < 1/4 + 1/2^2 + 1/4^3 + 1/4^4 + \ldots = 1/3 \). Hence it follows that the partitioning of the interval \([0, 1]\) into \( 2^r \) equal segments \( I_1, I_2, \ldots, I_{2^r} \) with \( r = 2 + 4 + \ldots + 2k \) for any arbitrarily large \( k \in \mathbb{N} \) contains \( t \) distinct segments on which \( f(x) \) takes the values 0 and 1 simultaneously, so that \( t \cdot 1/2^r > 2/3 \). Thus, the difference between the upper and lower Darboux sums for the function \( f \) on \([0, 1]\) is always less than \( 2/3 \). Q.E.D.

THEOREM 2. There exists a finite R-function \( \psi \) such that \( \psi \) is a continuous nowhere differentiable function.

Proof. The sought finite R-function \( \psi \) is the periodic function \( f = \psi \) on \([0, 1]\), and \( f \) is a polygonal line with nodes from the set \( M_1 \) with \( M_0 = \{(0, 0), (1/4, 1/2), (1/2, 1), (3/4, 1/2), (1, 0)\} \). Now, \( M_1 \subseteq M_{i+1} \) and if \( z_1, z_2 \) is a segment of the polyline \( \tilde{f} \) that contains no points from \( M_i \) other than \( z_j = (x_j, y_j), j = 1, 2 \), then \( M_{i+1} \) contains the points \( z_1 = [(x_1 + x_2)/2, y_2], z_2 = [(x_1 + y_2)/2, y_1] \) and the midpoints of the segments \( [z_1, z_2], [z_1', z_2'], [z_2', z_2] \). The nature of the bends in the polyline immediately implies nondifferentiability at all points. It is easy to see that the function \( \psi \) is defined on the set \( 0^\infty \) by the following three-state OR-transducer:

\[
\begin{align*}
q_0 & \rightarrow 0^\infty, \quad q_0 00 \rightarrow 0q_1, \quad q_0 01 \rightarrow 1q_1, \\
q_0 10 & \rightarrow 1q_2, \quad q_0 11 \rightarrow 0q_2, \\
q_2 & \rightarrow 0q_3, \quad q_2 00 \rightarrow 0q_3, \quad q_2 01 \rightarrow 1q_2, \quad q_2 10 \rightarrow 0q_3, \quad q_2 11 \rightarrow 1q_1, \\
q_1 & \rightarrow 0q_3, \quad q_1 00 \rightarrow 0q_3, \quad q_1 01 \rightarrow 1q_2, \quad q_1 10 \rightarrow 0q_3, \quad q_1 11 \rightarrow 1q_1, \\
q_2 & \rightarrow 1q_2, \quad q_2 00 \rightarrow 1q_2, \quad q_2 01 \rightarrow 0q_3, \quad q_2 10 \rightarrow 0q_3, \quad q_2 11 \rightarrow 0q_3.
\end{align*}
\]

†The case \( |\alpha| \leq 1 \) corresponds to an R-transducer command [1].