The paper considers representation of Hamiltonian circuits by natural arithmetic graphs.

Optimal representation of cycles in arithmetic graphs

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The notion of arithmetic graph was first introduced in [1] and subsequently generalized in a number of publications. The notion of numerical graph is defined in [2].

Definition 1. A numerical graph is the triple G = (X, U, F), where X = {x₁, x₂, ..., xₙ} is the set of real numbers defining the vertices, U = {u₁, u₂, ..., uₘ} is the set of real numbers called the generator set, and F is an analytical symmetric function of two variables. Two vertices xᵢ and xⱼ are adjacent if F(xᵢ, xⱼ) ∈ U.

An arithmetic graph is a numerical graph with F(xᵢ, xⱼ) = xᵢ + xⱼ. Among all possible arithmetic graphs, natural graphs with X = {1, 2, ..., n} are of special interest. These graphs are the subject of our study (in what follows, an arithmetic graph is always a natural graph).

Consider the matrix A = (ₐᵢⱼ), where

\[ aᵢⱼ = i + j; \quad aᵢᵢ = 0 \quad (1 ≤ i, j ≤ n). \]  

This matrix corresponds to an arbitrary complete arithmetic graph. We call it the generator matrix:

\[
A = \begin{pmatrix}
0 & 3 & 4 & 5 & \ldots & n & n + 1 \\
3 & 0 & 5 & 6 & \ldots & n + 1 & n + 2 \\
4 & 5 & 0 & 7 & \ldots & n + 2 & n + 3 \\
5 & 6 & 7 & 0 & \ldots & n + 3 & n + 4 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 & 2n - 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 2n - 1 & 0
\end{pmatrix}
\]  

We assume that the set U is irredundant, i.e., for any u ∈ U there is always a pair of vertices xᵢ and xⱼ (xᵢ ≠ xⱼ) such that xᵢ + xⱼ = u. The set U is assumed to be naturally ordered.

One of the main problems in the theory of arithmetic graphs is finding for a given graph the set U of minimum cardinality. Alongside this problem, it is relevant to consider the enumeration of all minimum-cardinality generator sets. The matrix A uniquely represents a complete n-vertex graph whose generator set is a segment of the natural series \(U = \{3, 4, 5, \ldots, 2n - 1\}\). Any proper subset of the set U defines a graph in which the number of edges multiplied by two equals the number of elements of the matrix A (because one edge \((xᵢ, xⱼ)\) is specified by two elements \(aᵢⱼ\) and \(aⱼᵢ\) that correspond to the given subset. For each u ∈ U we can count the number \(ϕ(u)\) of the corresponding elements of the matrix A. The largest value \(ϕ(u)\) is achieved for \(u = n + 1\) and these elements form the second diagonal of the matrix A. It is easy to note that \(ϕ(u)\) is symmetric relative to the value \(u = n + 1\), i.e., we have the equality

\[ ϕ(n + 1 - Δ) = ϕ(n + 1 + Δ) \quad (0 ≤ Δ ≤ n). \]  

If the arguments of the function are denoted by \( u \), we obtain

\[
\Delta = |n + 1 - u|,
\]

\[
\varphi(u) = \varphi(n + 1 - |n + 1 - u|).
\]

(4)

The function thus needs to be evaluated only for \( 1 \leq u \leq n + 1 \). It is easy to see that

\[
\varphi(u) = \begin{cases} 
  u - 2 & \text{for even } u; \\
  u - 1 & \text{for odd } u,
\end{cases}
\]

(5)

or

\[
\varphi(u) = 2 \left\lfloor \frac{u - 1}{2} \right\rfloor,
\]

(6)

where \( \lfloor \cdot \rfloor \) is the whole part of the expression in brackets. Substituting this value in (4), we finally obtain

\[
\varphi(u) = 2 \left\lfloor \frac{n - |n + 1 - u|}{2} \right\rfloor.
\]

(7)

Taking various values of \( u \) for the generator set \( U \), we can count the number of edges in the resulting graph and determine the degrees of its vertices. Two generators \( u_1 \) and \( u_2 \) correspond to two adjacent edges only if \( |u_1 - u_2| < n \).

Optimal representations of chains and the enumeration of all generator sets were fully described in [3], where some results about cycles were also obtained. In this paper, we present a complete description and enumeration of all optimal representations of cycles of length \( n \), which are denoted by \( C_n \).

**Definition 2.** The generator set \( U' = \{u_1', u_2', \ldots, u_m'\} \) is called dual relative to the set \( U = \{u_1, u_2, \ldots, u_m\} \) if its elements in the matrix \( A \) are symmetric about the second diagonal to the elements corresponding to the set \( U \).

A dependence between the sets \( U \) and \( U' \) is easily established. We clearly have \( (U')' = U \).

**Lemma 1.** There is a one-to-one correspondence between the elements of the dual sets \( U \) and \( U' \), \( U = 2n + 2 - u \).

We will show that \( \varphi(u) = \varphi(u') \) and thus prove the lemma:

\[
\varphi(u') = \varphi(2n + 2 - u) = \\
= 2 \left\lfloor \frac{n - |n + 1 - (2n + 2 - u)|}{2} \right\rfloor = \\
= 2 \left\lfloor \frac{n - |u - n - 1|}{2} \right\rfloor = \\
= 2 \left\lfloor \frac{n - |n + 1 - u|}{2} \right\rfloor = \varphi(u).
\]

In what follows, dual sets are taken into account automatically when enumerating the generator sets.

**Lemma 2.** For any cycle \( C_n \) there always exist two generators \( u_1 \leq n + 1 \) and two generators \( u_2 \geq n + 1 \).

Since each vertex in a cycle is of degree 2, the generators joining vertex 1 with two other vertices should not exceed \( n + 1 \). Similarly, vertex \( n \) is joined with two vertices and it requires two generators not less than \( n + 1 \).

**Lemma 3.** Three generators are sufficient for a cycle \( C_n \) with even \( n \).

From the previous lemma it follows that in this case we necessarily have \( u_2 = n + 1 \), \( u_1 < n + 1 \), \( u_3 > n + 1 \). Since each vertex in a cycle is of degree 2, the generators necessarily correspond to precisely two elements of the matrix \( A \).