ITERATIVE REGULARIZATION FOR SEMIINFINITE OPTIMIZATION

PROBLEMS

N. M. Novikova

UDC 519.85

Coordinated parameters for iterative regularization of the penalty method with infinitely many constraints are derived. A regularization algorithm for the mixed penalty and (stochastic) quasigradient method is considered.

1. Semiinfinite optimization problems are mathematical programming problems with infinitely many constraints, i.e., problems that seek the value $F^0$ and the realization $x^0 \in X$ such that

$$\min_{x \in X} F(x), \quad X = \{x \in X' \mid g(x, y) \leq 0 \quad \forall y \in Y\},$$

where $X'$ and $Y$ are compacta in the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, $F$ and $g$ are continuous and Lipschitzian functions: $F$ in $x$ with the constant $l$ and $g$ in $y$ with the constant $L \forall x \in X', X \neq \emptyset$. (Under these conditions, the minimum in (1) is achieved.) Such problems arise in connection with approximation of infinite-dimensional constrained optimization problems, which accounts for their name.

The infinity of constraints in problem (1) may be reduced, similarly to [1], to a single integral constraint

$$G^0(x) = \int \left\{ \max_{y} \{0, g(x, y)\} \right\} dy = 0, \quad (\eta \geq 1),$$

which is then eliminated by the penalty method,

$$F^0 = \lim_{t \to \infty} \min_{x \in X'} \{F(x) + C_t G^0(x)\}, \quad \forall C_t \uparrow \infty.$$

Denoting by $a^t$ an arbitrary realization of the minimum in (3), by $X^0$ the solution set of problem (1), and by $\rho$ the distance in $\mathbb{R}^n$, we obtain for the integral penalty (2), similarly to [1, 2],

$$\rho(a^t, X^0) \to 0 \quad \text{for} \quad C_t \uparrow \infty.$$  

Unfortunately, (4) does not imply convergence of the sequence $\{a^t\}$ to some solution $x^0 \in X^0$, which is inconvenient for practical computation. In order to construct a sequence that converges to a solution, we perform iterative regularization of the penalty method (3) similarly to [3].


0011-4235/91/2701-0115$12.50 © 1991 Plenum Publishing Corporation
Note that the integral penalty $G^q$ may be smooth even for $q = 1$, and then the equality in (3) is not attained for any finite $C_t$ or for all $C_t \geq 0$ (and the constraints $X$ are not bindings). The assumptions of the existence of a saddle point of the Lagrange function used in [3] (which implies finiteness of $C_t$ in (3) [2, p. 112]) is therefore too restrictive for problem (1). We thus propose a weaker condition.

**Condition 1 (regularity of constraints $X$):**

$$\exists K, \varepsilon > 0, \theta > 1: \forall x' \in \{x \in X \setminus X \mid \rho(x, X) \leq \varepsilon\} \ G^q(x') \geq K \{\rho(x', X)\}^\theta.$$ 

A similar condition with $\theta > 0$ is introduced in [2] (condition b of Theorem 1.4 — rate of convergence bounds of the penalty method with finitely many constraints), but only the case $\theta = 1$ is considered as a regularity condition in [2, p. 54]. This case, however, is not characteristic for the constraints (2), because it assumes (implies by Theorem 1.4 of [2, p. 51]) a finite penalty, as mentioned above.

If a finite penalty exists, then the regularization parameters and the penalty parameters do not require coordination, and convergence of the iterative regularization in the mathematical programming problem without iterative penalties of the constraints was proved, e.g., in [4]. It was also shown in [4] that the gradient method has self-regularizing properties, and therefore in many problems sequences convergent to a solution may be constructed by projections of gradients (conditional, reduced, and other gradients). But removal of infinitely many constraints, unlike penalizing, is an independent problem, and regularization is therefore an essential component of problem (1), especially for $\theta > 1$.

Tikhonov's function [5, 3] of our problem has the form

$$T_t(x) = F(x) + C_tG^\theta(x) + \alpha_t\Omega(x), \quad \alpha_t \downarrow 0,$$

where the regularizer $\Omega(.)$ is a strongly convex function in $\mathbb{R}^n$. Denote by $b^t$ the (unique) realization of the minimum of Tikhonov's function on $X'$. By condition 1, we coordinate the increase of $C_t$ and the decrease of $\alpha_t$:

$$\exists \alpha > 0, \quad \alpha_t \geq \alpha \left( \frac{1}{C_t} \right)^{1/\theta}, \quad \forall t = 1, 2, ..., \quad \alpha_t \downarrow 0, \quad t \to \infty.$$ 

**THEOREM 1.** Under the above assumptions, given condition 1 and coordination (6), the sequence $b^t$ converges to the $\Omega$-normal solution $b \in X^0$ of problem (1), i.e., to a (unique) realization $b$:

$$\min_{x \in X^0} \Omega(x), \quad X^0 = \text{arg min}_{x \in X} F(x).$$ 

**Proof.** By definition $T_t(b^t) \leq T_t(b)$, i.e.,

$$0 \leq C_tG^\theta(b^t) \leq F(b) - F(b^t) + \alpha_t \{\Omega(b) - \Omega(b^t)\}. \quad (7)$$

From the conditions of problem (1) the right-hand side of (7) is bounded, and therefore $G^\theta(b^t) \to 0$ as $C_t \uparrow \infty$, and any limiting point $b'$ of the sequence $b^t$ satisfies $b' \in X$. Passing in (7) to the limit as $t \to \infty$ and using $b \in X^0$, we obtain

$$0 \leq \lim C_tG^\theta(b^t) \leq F(b) - F(b^t) \leq 0.$$ 

Hence $F(b^t) = F^0$, i.e., $b' \in X^0$ and $C_tG^\theta(b^t) \to 0$ as $t \to \infty$.

Denote $M = \{t = 1, 2, ..., \mid F(b^t) < F^0\}$. Clearly $\forall t \in M \ b^t \in X \setminus X$ and starting with sufficiently large $t \in M$, $\rho(b^t, X) \leq \varepsilon$, which by condition 1 implies $K\rho^\theta(b^t, X) \leq G^\theta(b^t) = o(1/C_t)$ from the previously proved facts. Let $c^t$ be the element of the set $X$ nearest to $b^t$, i.e., $\rho(c^t, X) = \|c^t - b^t\|$. Then $\forall t \in M \ 0 < F^0 = F(b^t) \leq F(c^t) + \alpha_t \{\Omega(b) - \Omega(b^t)\} \leq l\|c^t - b^t\| = l\rho(b^t, X) \leq l \{G^\theta(b^t)\}^{1/\theta} = o(C_t)^{-\theta}$, where $l$ is the Lipschitz constant of the function $F$. Rewrite (7) in the form

$$\Omega(b') - \Omega(b) \leq \frac{F^0 - F(b^t)}{\alpha_t} \leq ...$$