DIFFUSION APPROXIMATION OF VIRTUAL WAITING TIME IN M/M/1
SYSTEM (MARTINGALE APPROACH)

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Limit theorems are proved for virtual waiting time under high traffic conditions, using martingale methods.

In this paper, we consider a single-server queueing system with Poisson arrivals $A = (A_t)_{t=0}$. The service times $(\eta_i)_{i\geq1}$ constitute a sequence of independent identically distributed random variables with $E\eta_1 = \mu^{-1}$. Denote by $Q_t$ the queue length at time $t$.

The virtual waiting time process $V = (V_t)_{t>0}$ is defined by the equation

$$V_t = \sum_{i=1}^{Q_{t+N_t}} \eta_i - \int_0^t I(V_s > 0) \, ds.$$

The main result of this study is the proof of limit theorems for the virtual waiting time under high traffic conditions. The problem is implemented using the martingale methods of [1].

Note that the same methods were previously applied in [2] to prove a limit theorem which shows that under high traffic conditions the normalized queue weakly converges in a certain sense to an Ornstein-Uhlenbeck process. The virtual waiting time under high traffic conditions was studied by various methods in [3-4]. A bibliography on other approaches to the diffusion approximation in queueing theory is given in [2].

1. BASIC ASSUMPTIONS AND NOTATION

Consider the series scheme, where for each $n \geq 1$:
1) $Q^n_t$ is the queue length at time $t$ in series $n$ ($n = 1, 2, \ldots$) and $Q^n_0 = 0$;
2) the arrival process $A^n = (A^n_t)_{t>0}$ is Poisson with the parameter $n\lambda$;
3) service times $(\eta^n_i)_{i\geq1}$ form a sequence of independent random variables with $E\eta_1^n = 1/n\mu$, $E(\eta_1^n - 1/n\mu)^2 = \sigma^2/n^2$, where $\mu$ and $\sigma^2$ are positive constants;
4) the random objects $A^n$ and $(\eta_i^n)_{i\geq1}$ are assumed independent;
5) we assume that $\lambda \geq \mu$;
6) $\lim_n E(\eta_1^n)^2 I(\eta_1^n > \varepsilon\sqrt{n}) = 0$.

Remark. The notation $\rightarrow^p$ denotes convergence in probability and $\rightarrow^d$ denotes weak convergence of distributions of random variables and processes (in the latter case, in the Skorokhod topology of the space $D_{[0,\infty)}$).

2. VIRTUAL WAITING TIME

The virtual waiting time for each $n$, allowing for $Q_0^n = 0$, is given by

$$V^n_\tau = \sum_{i=1}^{A^n_i} \eta^n_i \int_0^\tau I(V^n_s > 0) \, ds.$$  \hfill (1)

We will show that for $h \to \infty$ the sequence of processes $V^n = (V^n_t)_{t \geq 0}$ converges to a limit.

**THEOREM 1.** Assume that conditions 2-5 are satisfied. Then for any $T > 0$

$$\sup_{t \leq T} \left| V^n_t - \left( \frac{\lambda}{\mu} - 1 \right) t \right| \mathop{\to}^p 0.$$  \hfill (2)

**Proof.** Let

$$H^n_\tau = \sum_{i=1}^{A^n_i} \eta^n_i, \quad \hat{V}^n_\tau = H^n_\tau - t$$  \hfill (3)

and note that $V^n_t$ is the solution of the equation

$$V^n_\tau = \hat{V}^n_\tau + \int_0^\tau I(V^n_s = 0) \, ds.$$  \hfill (4)

Equation (3), as is well known, has the solution

$$V^n_\tau = \hat{V}^n_\tau - \inf_{s \leq \tau} (\hat{V}^n_s \wedge 0).$$  \hfill (5)

From (4) we obtain two bounds:

$$\sup_{t \leq T} |V^n_t - \hat{V}^n_t| \leq \inf_{s \leq T} (\hat{V}^n_s \wedge 0),$$  \hfill (6)

From (5) it follows that the theorem holds if

$$\sup_{t \leq T} \left| \hat{V}^n_\tau - \left( \frac{\lambda}{\mu} - 1 \right) t \right| \mathop{\to}^p 0,$$  \hfill (7)

and it is therefore sufficient to prove (7). To this end, define the family of $\sigma$-algebras $\mathcal{F}_t = (\mathcal{F}_t^n)_{t \geq 0}$ with $\mathcal{F}_t^n = \mathcal{F}_t^{H^n}$ generated by the process $H^n = (H^n_t)_{t \geq 0}$ (completed and right-continuous).

We will need the following auxiliary result.

**LEMMA 1.** A. The compensator $\hat{H}^n = (\hat{H}^n_t)_{t \geq 0}$ of an increasing process $H^n$ relative to $\mathcal{F}_t^n$ is computed by the formula $\hat{H}^n_t = (\lambda/\mu)t$. 

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