A PURSUIT GAME WITH MOVING OBJECTS

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The player P (the pursuer) and the player E (the pursued or the evader) move within a circle of radius R. Player P moves with constant speed and can change the direction of his motion at any instant; player E can move completely freely. At any instant, P has no information about the position and velocity of E, while E can know only the speed at which P moves. At any time t both players have complete information about their own motion from the beginning of the game up to time t. It is P's object to capture E in minimum time; this is the pay-off in this game. Of course E's objective is the opposite one.

It will be assumed that capture has occurred if the distance PE is zero.

Let us denote the velocity and position of P at time t by \( u(t) \) and \( x_t \), and the velocity and position of E at time t by \( v(t) \) and \( y_t \). Without loss of generality we can assume that \( x_0 = 0, y_0 = l \), where \( 0 < l < 2\pi R \). Thus at the beginning of the calculation a point is chosen inside the circle and the layer P is put there at the time the game starts. The positive direction of motion is taken to be that opposite to the direction of the hour hand.

Since the game we have described is obviously a game with incomplete information, it is natural to assume that \( u(t) \) and \( v(t) \) are independent random processes. Therefore we shall say that the strategies of the player P are the set of all possible compatible finite dimensional distributions of the process \( u(t) \)

\[
F_{i_1, i_2, \ldots, i_n}(u_1, u_2, \ldots, u_n), \quad i_i \geq 0, \quad u_i = \pm u, \quad i = 1, n, \quad n = 1, 2, \ldots
\]  

(1)

Similarly the set of all possible compatible finite dimensional distributions of the process \( v(t) \)

\[
G_{i_1, i_2, \ldots, i_n}(v_1, v_2, \ldots, v_n), \quad i_i \geq 0, \quad -\infty < v_i < \infty
\]  

(2)

we shall call the strategies of the player E.

Let \( T \) be the instant at which the players meet. The mean of this time of meeting, \( M_T \), depends on the strategies \( F \) and \( G \) chosen by the players, i.e.,

\[
M_T = M(F, G)
\]  

(3)

where \( F \) and \( G \) denote the fixed sets of finite dimensional distributions.

We must find \( F_0 \) and \( G_0 \) for P and E such that for any \( F \) and \( G \) the following inequality holds:

\[
M(F_0, G) \leq M(F, G_0) \leq M(F, G_0).
\]  

(4)

Partition the time interval \([0, \infty)\) into intervals of length \( \tau = \pi R / u \) and call them cycles. The following theorem results.

THEOREM. Among the sets of strategies of player P there is a strategy such that for any strategy of player E the mean number of cycles from the start up to the meeting is less than or equal to two. For player E there is a strategy such that for any strategy of player P the time to the meeting has mean greater than or equal to \( \tau \). If the pay-off \( M_T \) for the game exists, then it satisfies:

\[
\frac{\pi R}{u} \leq M_T \leq \frac{2\pi R}{u}.
\]  

(5)

Proof. Consider player P’s strategy: with probability $p_i$, he moves from the starting point against
the hour hand for a time $\tau$ and with probability $1-p_i$ he moves along the hour hand for a time $\tau$. If a
meeting does not occur during the time $\tau$, player P repeats this cycle with the probabilities $p_i$ and $1-p_i$,
respectively. This process is continued until a meeting occurs.

Let $\mathbf{p}^{(n)}$ be the probability that the meeting first occurs in the $n$-th cycle. We then have

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n)} (A_n \cap \overline{A}_{n-1} \cap \ldots \cap \overline{A}_1) = \mathbf{p} (A_n / \overline{A}_{n-1} \cap \ldots \cap \overline{A}_1) \cdot \mathbf{p} (\overline{A}_{n-1} / \overline{A}_{n-2} \cap \ldots \cap \overline{A}_1) \ldots \mathbf{p} (\overline{A}_1),$$

where $A_1$ is the event that the meeting occurs in the $1$-th cycle, $1=1,2,\ldots,n$, $n=1,2,\ldots$.

The probability that the meeting occurs in the $i$-th cycle can be written in the form

$$\mathbf{p}_i = p_i \mathbf{g}_i + (1-p_i) \mathbf{g}_i^*$$

where $p_i$ is the probability that in the $i$-th cycle player P moves against the hour hand, $\mathbf{g}_i$ is the probability
that the meeting occurs in the $i$-th cycle under the hypothesis that player P moves against the hour hand,
and $\mathbf{g}_i^*$ is the probability that the meeting occurs in the $i$-th cycle under the hypothesis that P moves along
the hour hand. We now prove that for any $i=1,2,\ldots$ we have

$$1 \leq \mathbf{g}_i + \mathbf{g}_i^* \leq 2.$$

Let us denote the set of trajectories which leave the point $y_0 = I$ and which satisfy, for $0 \leq t \leq \tau$, the
inequality

$$u_t < y_t < u_t + 2\pi R,$$

by $Y_t^+$, and let us denote the set of trajectories which start at $I$ and satisfy the inequality

$$-u_t < y_t < 2\pi R - u_t$$

for $0 \leq t \leq \tau$, by $Y_t^-$. The continuity of the trajectories $y_t$ and (8) and (9) give us:

$$\{Y_t^+\} \cap \{Y_t^-\} = \emptyset,$$

and therefore

$$\mathbf{p} (Y_t^+ \cup Y_t^-) = \mathbf{p} (Y_t^+) + \mathbf{p} (Y_t^-).$$

Since

$$\mathbf{p} (Y_t^+) = 1 - \mathbf{g}_t^*, \mathbf{p} (Y_t^-) = 1 - \mathbf{g}_t^*,$$

it follows from (10) that (7) holds for $i=1$.

The proof for arbitrary $i$ is similar.

It can be shown that the mean of the number of cycles up to the first meeting is

$$MN = \sum_{i=1}^{\infty} \mathbf{p}^{(i)} = \sum_{i=1}^{\infty} (1 - p_{i-1}) \cdots (1 - \mathbf{p}),$$

where $\mathbf{P}_0 = 0$.

If we look at $\mathbf{p}_i, i=1,2,\ldots$, for the various $p_i, \mathbf{g}_i, \mathbf{g}_i^*$ we have from (8) that

$$\min_{i} \mathbf{p}_i = \min (p_i, 1-p_i),$$

and therefore when $p_i = 1/2$ we get

$$\frac{1}{2} \leq \mathbf{p}_i \leq \frac{1}{2} (\mathbf{g}_i^* + \mathbf{g}_i).$$