1. Economic Model of One Country

In one-product models of economic growth two factors of production, i.e., labor and basic capital, are considered [1]. Let $K(t)$ and $L(t)$ denote, respectively, the size of the basic capital funds and the labor resources at time $t$, $t \geq 0$. Then, the quantity of pure homogeneous product $Y(t)$ prepared in unit time by labor $L(t)$ with the use of basic funds $K(t)$ is defined as

$$Y(t) = F(K(t), L(t)).$$  \hfill (1)

where function $F(\cdot)$ is called productivity.

At each moment of time, part of the fruits of production are directed towards consumption, while the remaining portion is devoted to an increment of basic funds. We denote by $C(t)$ and $Z(t)$, respectively, the usage and accumulation of product in unit time, so that then

$$Y(t) = C(t) + Z(t).$$  \hfill (2)

If it is assumed that depletion of basic funds occurs exponentially, with exponent $\mu > 0$, i.e., if we set

$$K(t) = K(0) e^{-\mu t} + \int_{0}^{t} Z(t) e^{-\mu(t-t)} dt,$$  \hfill (3)

then the equation describing the change of basic funds has the form

$$\frac{dK(t)}{dt} = Z(t) - \mu K(t).$$  \hfill (4)

We fix the initial amount of basic funds $K(0) = K_0$, and we shall assume as known the growth function of labor resources $L(t)$. Then, with a specified rule for distribution of product

$$C(t) = R(t, Y(t))$$  \hfill (5)

we obtain, from the joint solution of Eqs. (1), (2), (4), and (5), $K(t)$, $Y(t)$, $C(t)$, and $Z(t)$ for any $t \geq 0$.

In the present paper we shall assume that, at each moment of time, a definite fraction of finished product is singled out for use, i.e.,

$$C(t) = (1 - \gamma) Y(t), \quad 0 < \gamma < 1.$$  \hfill (6)

and we shall be interested in an economy in which, in the time interval under consideration $0 \leq t \leq T$, the output of product depends linearly on the size of the basic funds, i.e.,

$$Y(t) = F(K(t), L(t)) = a \cdot K(t), \quad a > 0.$$  \hfill (7)

For the parameters $\mu$, $\gamma$, and $a$ we adopt the relationship

$$\gamma \cdot a > \mu,$$  \hfill (8)

and then the volume of basic funds, output, usage, and accumulation over the interval $0 \leq t \leq T$ grow exponentially at the same tempo $a = \gamma \cdot a - \mu$.

2. Model of Two-Country Cooperation

We now take up the study of possibilities of cooperation between two countries. We shall assume that the economy of each country is described by the model already considered, (1)-(8). The same economic characteristic appertaining to different countries will be distinguished by the appropriate subscript, \( i = 1, 2 \). Cooperation consists in the facility of, at each moment of time, permitting some part of the productive output of one country to be transferred to the other. If we denote by \( U(t) \) the amount of the transmission then we adopt the convention that \( U(t) > 0 \) when the production is sent to the second country, while \( U(t) < 0 \) in the opposite case.

We consider two assertions concerning the transmitted production \( U(t) \).

1. The transferred product was destined for increasing basic funds, and it is thus used in its new locus. Hence, we obtain for \( U(t) \) restrictions of the form

\[
\begin{align*}
\gamma_i Y_i - U &\geq 0, \\
\gamma_i Y_i + U &\geq 0,
\end{align*}
\]

and, for the change in amount of funds, the equations

\[
\begin{align*}
\dot{K}_1 &= (\gamma_i Y_i - U) - \mu_i K_1 = a_i K_1 - U, \\
\dot{K}_2 &= (\gamma_i Y_i + U) - \mu_i K_2 = a_i K_2 + U.
\end{align*}
\]

2. All product produced is allowed to be transferred, and the allocation to usage and accumulation applies, not to output \( Y_1 \) and \( Y_2 \), but to the corresponding volumes of product \( Y_1 - U \) and \( Y_2 + U \). In this case we have

\[
\begin{align*}
Y_1 - U &\geq 0, \\
Y_2 + U &\geq 0
\end{align*}
\]

and

\[
\begin{align*}
\dot{K}_1 &= \gamma_1 (Y_1 - U) - \mu_i K_1 = a_1 K_1 - \gamma_1 U, \\
\dot{K}_2 &= \gamma_2 (Y_2 + U) - \mu_i K_2 = a_2 K_2 + \gamma_2 U.
\end{align*}
\]

In the sequel we shall also be interested in the conditions of cooperation under which, in the course of time, the amount of basic funds of neither country is decreased, i.e., \( \dot{K}_1 \geq 0 \) and \( \dot{K}_2 \geq 0 \). This leads to stronger constraints on the admissible amount of transfer \( U \), both in the first and in the second case. We note further that the change of variables \( K'_1 = (1/\gamma_i) K_1 \), \( i = 1, 2 \), brings system (10) to the form of Eqs. (9).

Therefore, in the sequel we shall embark upon the consideration of the economic systems of two countries whose dynamics are described by the differential equations

\[
\begin{align*}
\frac{dK_1(t)}{dt} &= a_1 K_1(t) - U(t), \\
\frac{dK_2(t)}{dt} &= a_2 K_2(t) + U(t), \quad t \geq 0,
\end{align*}
\]

for given initial state

\[
(K_1(0), K_2(0)) = (K_1^0, K_2^0)
\]

and with constraints on \( U(t) \) of the form

\[
\begin{align*}
a_1 K_1(t) - U(t) &\geq 0, \\
a_2 K_2(t) + U(t) &\geq 0.
\end{align*}
\]

For definiteness, we set

\[
a_1 > a_2
\]

The ultimate goal of the study of cooperation will be the construction of set \( D_T \) of states which can be attained in time \( T, T > 0 \). State \( K^1 = (K_1^1, K_2^1) \) is said to be attainable if there exists an agreement on cooperation \( U(t), 0 \leq t \leq T \) satisfying relationship (13)], which provides for the transition of the economic systems from initial state \( (K_1^0, K_2^0) \) to the final state \( (K_1(T), K_2(T)) = K^1 \). By virtue of the linearity of Eqs. (11) and inequalities (13), set \( D_T \) is convex, so that, to determine its form, it suffices to obtain a solution to the following problems.