DIFFERENTIABILITY OF A FUNCTIONAL OF A PROGRAMMING
PROBLEM IN AN INFINITE-DIMENSIONAL SPACE

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INTRODUCTION

In solving certain minimax or minimum problems, one must deal with a function of the following
special form:

\[ F(y) = \sup_{x \in \mathcal{E}} \varphi(x, y). \]

It is necessary to give an expression for the directional derivative* and, if the derivative is convex, to
describe the set of support functionals. The known results are contained in [1, 2, 3].

If \( \varphi(x, y) \) is continuously differentiable in \( y \) and continuous in \( x \), and if \( \mathcal{Q} \) is compact, then the
directional derivative along \( \tilde{y} \) has the form

\[ F'_y(g_\tilde{y}; \tilde{y}) = \max_{x \in M_0} \varphi'_y(x, y; \tilde{y}) \tag{1} \]

(see [1, 2]), where \( M_0 = \{x/x \in \mathcal{Q}, \varphi(x, y_0) = F(y_0)\} \).

Suppose that \( \mathcal{Q} \) is either an arbitrary compact set [2], or a compact set in a finite-dimensional space
[3]. Suppose that \( \varphi(x, y) \) is continuous in \( x \) and \( y \), convex in \( y \), and has a unique support function \( \mu_x \) for every
\( x \in \mathcal{Q} \), and, moreover, that \( \mu_x^{\ast} = \mu_{x_0} \) as \( x \to x_0 \). Then the set \( M(y_0) \) of support functionals to \( F(y) \) at \( y_0 \)
is given as the weak closure of the convex hull of \( M_0 \): \( M(y_0) = \overline{\text{conv}}(M_0) \), where

\[ M = \{\mu_x/\mu_x \in Y^\ast, \varphi(x, y) - \varphi(x, y_0) \geq \mu_x(y - y_0), x \in \mathcal{Q}\} \]
or, in an explicit form,

\[ M(y_0) = \left\{ \mu/\mu(y) = \int_{\mathcal{M}_0} \mu_x(y) \, d\omega(x), \int_{\mathcal{M}_0} d\omega = 1 \right\} \]

and

\[ F'_y(g_\tilde{y}; \tilde{y}) = \max_{\mu \in M_0} \mu(\tilde{y}). \tag{2} \]

In practical applications, our assumptions are very restrictive, though results of the form (1) and (2),
for example, hold, as we shall show, under weaker assumptions.

In this paper we will investigate, in as general a formulation as possible, the problems of directional
differentiability of the function \( F(y) \) and of constructing the set of support functionals to it.

1. BASIC RESULTS

Let \( \varphi(x, y) \) be a real-valued function, let \( \mathcal{Q} \) be an arbitrary set of the elements, let \( Y \) be a B-space,
and let \( F(y) = \sup_{x \in \mathcal{Q}} \varphi(x, y), y \in Y \). The set \( U(y_0) \) is a neighborhood of \( y_0 \in Y \).

*The limit \( \lim_{\epsilon \to 0} \frac{F(y + \epsilon \tilde{y}) - F(y)}{\epsilon} \) is called the directional derivative of the function \( F(y) \) at point \( y \)
along a direction \( \tilde{y} \in Y \).
Lemma 1. The function $F(y)$ is differentiable along every direction $\tilde{y} \in Y$ at $y_0 \in Y$, if:

1) The function $\varphi(x, y_0)$ is bounded on $Q \times U(y_0)$ and differentiable along every direction $\tilde{y} \in Y$, and moreover, $\varphi'_i(y_0; \tilde{y})$ is bounded on $Q \times U(y_0)$.

2) For every $\tilde{y} \in Y$, there exists the limit

$$
\lim_{\delta \to 0} \sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon},
$$

where

$$
M_\delta = \{x \mid x \in Q, \varphi(x, y_0) > F(y_0) - \delta, \delta > 0\}.
$$

Under these conditions

$$
F'_i(y_0; \tilde{y}) = \lim_{\delta \to 0} \sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon}. \tag{1.1}
$$

Proof. Let us bound the quotient $F(y_0 + \varepsilon \tilde{y}) - F(y_0) / \varepsilon$, $\varepsilon > 0$ from both the left and right sides:

$$
\frac{F(y_0 + \varepsilon \tilde{y}) - F(y_0)}{\varepsilon} \geq \sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon}, \tag{1.2}
$$

$$
\lim_{\delta \to 0} \sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon}.
$$

We shall show that, for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the following bound on the left holds:

$$
\sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon} \geq F(y_0 + \varepsilon \tilde{y}) - F(y_0). \tag{1.3}
$$

Let an $\varepsilon_0 > 0$ be such that $y_0 + \varepsilon \tilde{y} \in U(y_0)$, $0 \leq \varepsilon \leq \varepsilon_0$. For an arbitrary $0 \leq \varepsilon \leq \varepsilon_0$, we choose a $\delta(\varepsilon)$ from the inequality

$$
\varepsilon \left( \sup_{\delta \in [\varepsilon_0, \varepsilon]} \sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon} \right) \leq F(y_0) - \delta. \tag{1.4}
$$

Inequality (1.4) can be solved for $\delta(\varepsilon)$ with $0 \leq \varepsilon \leq \varepsilon_0$, since the quantity in the parentheses is, by virtue of Condition 1 of Lemma 1, nonnegative and bounded. Letting $\varepsilon$ tend to zero, we can make the left-hand side arbitrarily small independently of $\delta$. Thus, we can construct $\delta(\varepsilon) \to 0$ such that (1.4) still holds for $0 \leq \varepsilon \leq \varepsilon_0$. We let $0 \leq \varepsilon \leq \varepsilon_0, x \in M_\delta$, and $x \in Q$. Then $\varphi(x, y_0) \leq F(y_0) - \delta(\varepsilon)$ and

$$
\varphi(x, y_0) + \varepsilon \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon} \leq \varphi(x, y_0) + \varepsilon \sup_{\delta \in [\varepsilon_0, \varepsilon]} \sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon} \leq F(y_0) + \varepsilon \sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon},
$$

by virtue of (1.4). For $x \in M_\delta(\varepsilon)$ and $x \in Q$ this last inequality is obvious.

Thus, we obtain that

$$
\sup_{x \in \tilde{y}} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon} \leq F(y_0) + \varepsilon \sup_{x \in M_\delta(\varepsilon)} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon}. \tag{1.5}
$$

Inequality (1.3) follows from (1.5).

Taking into account (1.2) and (1.3), and applying condition 2 of Lemma 1 and the well-known theorem of analysis on double limits [4], we obtain the assertion (1.1) of the lemma.

Corollary. If $\varphi(x, y)$ is convex in $y$ for every $x \in Q$ and bounded on $Q \times U(y_0)$, then (1.1) holds.

Indeed, Condition 2 of Lemma 1 is satisfied for $\varphi(x, y)$ which is convex in $y$, since

$$
\sup_{x \in M_\delta} \frac{\varphi(x, y_0 + \varepsilon \tilde{y}) - \varphi(x, y_0)}{\varepsilon} = \frac{\varphi(x, y_0) + \varepsilon \tilde{y} \varphi'(x, y_0)}{\varepsilon} \to 0
$$

as $\delta \to +0$ and $\varepsilon \to +0$, and since it is bounded from below.

The results of Lemma 1 can be significantly sharpened. We assume that it is possible to introduce a topology in $Q$ such that from any sequence $x_i \in M_{\delta_i}, \delta_i \to 0, i = 1, 2, \ldots$, a subsequence can be chosen which