ON THE CALCULATION OF CERTAIN SINGULAR INTEGRALS

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A method for the approximate calculation of Cauchy-type integrals with logarithmic singularities is proposed. It is based on the expansion of a function $f(t)$ in a series in the Chebyshev polynomials.

For $x \in [-1; 1]$, we denote

$$I = I(f|_x) = \frac{1}{\pi} \int_{-1}^{1} \ln |t| \frac{f(t)}{t-x} \, dt.$$  \hspace{1cm} (1)

This integral, understood in the sense of the Cauchy principal value, is encountered in different problems of continuum mechanics. Many quadrature formulas exist for its approximate calculation [1]. Its expansion in a Chebyshev series is obtained in [2]. In this paper, we propose another method based on the assumption that the function $f(t)$ in (1) can be represented in the form of a series uniformly convergent on $[-1; 1]$

$$f(t) = \sum_{k \geq 0} a_k P_k(t),$$

where $P_k(t)$ are either the Chebyshev polynomials $T_k(t)$ of the first kind, or $P_k(t) = P_{2k}(t)$, or $P_k(t) = P_{2k+1}(t)$. For a segment

$$f_N(t) = \sum_{k=0}^{N} a_k P_k(t)$$  \hspace{1cm} (2)

of the series, we propose a method for calculating the corresponding integral $I_N = I(f_N|_x)$ similar to the well-known Klenshaw algorithm for finding $f_N(t)$ [3, 4].

We set

$$p_n(x) = \int_{-1}^{1} \ln |t| \frac{P_n(t)}{t-x} \, dt, \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (3)

It is known [1] that $p_0(x) = \frac{x^2}{2} \text{sign } x - 2N(x)$ for $P_0(t) = 1$, where

$$N(x) = \int_{0}^{x} \text{arcth} \frac{t}{t} \, dt, \quad \text{arcth} t = \frac{1}{2} \ln \frac{1+t}{1-t}.$$
In [5], one can find the representations of \( N(x) \) convenient for numerical computation. In addition, the Chebyshev coefficients of this function are presented. Note that \( N(x) \) appears in the explicit formulas obtained in [5] for computing integrals (3) with \( P_n = T_n \). At the same time, the value of

\[
I_N = \frac{1}{\pi} \sum_{k=0}^{N} a_k p_k(x)
\]

is found here without computing \( p_k(x), k \geq 1 \); it is only necessary to know \( p_0(x) \) and the coefficients \( a_k \) in equality (2). The approach of this sort is based on the difference equation

\[
p_{n+1}(x) + \alpha p_n(x) + p_{n-1}(x) = \gamma_n, \quad n \geq 1.
\]

In the case where \( P_n = T_n \), we have

\[
\alpha = -2x, \quad \gamma_n = \begin{cases} 
0, & n = 2r + 1, \\
4\sigma_r, & n = 2r.
\end{cases}
\]

If \( P_n = T_{2n} \), then \( \alpha = -2(2x^2 - 1) \) and \( \gamma_n = 8x\sigma_r \); finally, for \( P_n = T_{2n+1} \), we have \( \alpha = -2(2x^2 - 1) \) and \( \gamma_n = 4(\sigma_n + \sigma_{n+1}) \), where \( \sigma_n \) have the form

\[
\sigma_n = \int_0^1 \ln t T_{2n}(t) \, dt, \quad n = 0, 1, 2, \ldots.
\]

The simplest way of finding these numbers is to use the recurrent relation [2]

\[
\left(1 + \frac{1}{2k+2}\right)\sigma_{k+1} + \left(1 - \frac{1}{2k}\right)\sigma_k = \frac{1}{2k+1} \left[ \frac{1}{(2k+2)(2k+3)} - \frac{1}{(2k-1)2k} \right],
\]

where \( k \geq 1, \sigma_0 = -1, \) and \( \sigma_1 = 7/9. \)

Equality (5) can be easily derived from the definition of \( p_n(x) \) and the recurrent relations for the Chebyshev polynomials.

Let us describe the algorithm for computing (4) by using (5).

**Algorithm.** Given \( N \) and the coefficients \( a_0, \ldots, a_N \) in (2), we define the array \( \{b_0, \ldots, b_{N+2}\} \) by setting

\[
b_n = -\alpha b_{n+1} - b_{n+2} + a_n, \quad n = N, N-1, \ldots, 1, 0.
\]

Thus, for \( P_n = T_n \), we get

\[
\pi I_N = (b_0 - b_1 x)p_0(x) + \sum_{i=0}^{N-1} b_{i+1} \gamma_i;
\]

for \( P_n = T_{2n} \), we have

\[
\pi I_N = [b_0 - (2x^2 - 1)b_1]p_0(x) + \sum_{i=0}^{N-1} b_{i+1} \gamma_i;
\]

(6)