
SOLVABLE GROUPS OF FINITE NON-ABELIAN RANK

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The concept of the non-Abelian rank of a group is introduced. Solvable groups of finite non-Abelian rank are studied and it is proved that their (special) rank is finite.

1. Definition. Suppose G is a group and $\mathcal{F}$ a system of finitely generated subgroups. By the $\mathcal{F}$-rank of G we mean the smallest number r such that each subgroup in the system $\mathcal{F}$ can be generated by at most r elements. If there is no such number r, we say that the $\mathcal{F}$-rank of G is infinite.

Note that if $\mathcal{F}$ consists of all finitely generated subgroups of the group, then the concept of $\mathcal{F}$-rank agrees with that of the special rank introduced by Mal'tsev [1] and is usually called the rank of the group. On the other hand, finiteness of the general rank of a group [1] is equivalent to finiteness of the $\mathcal{F}$-rank for some local system of finitely generated subgroups of the group.

In the present paper, as in [2], we study groups of finite $\mathcal{F}$-rank, where $\mathcal{F}$ is the system of all non-Abelian, finitely generated subgroups. In this case, for brevity, the $\mathcal{F}$-rank of a group G will be called the non-Abelian rank of G and will be denoted by $\overline{r}(G)$.

As usual, $r(G)$ will be used to denote the special rank of G.

It was shown in [2] that finiteness of the non-Abelian rank $\overline{r}(G)$ implies finiteness of the rank $r(G)$, if G is a periodic, locally solvable group or a locally nilpotent, torsion-free group. Our main result is the following:

**THEOREM.** A solvable group of finite non-Abelian rank has finite rank.

This result was announced earlier in [3].

2. The proof of the theorem will be preceded by several auxiliary results.

**LEMMA 1.** If K is a non-Abelian normal subgroup of a group G having finite non-Abelian rank, then $r(G/K) \leq \overline{r}(G)$.

**Proof.** Suppose $H/K$ is any finitely generated subgroup of $G/K$. We represent H as a product $H = SK$, where S is some non-Abelian, finitely generated subgroup. It follows from
the definition of the non-Abelian rank of $G$ that the subgroup $S$ can be generated by at most $\bar{r}(G)$ elements. Therefore, in view of the relation $H = SK$, the factor group $H/K$ can also be generated by at most $\bar{r}(G)$ elements. This proves the desired relation $r(G/K) \leq \bar{r}(G)$.

**COROLLARY.** If a group $G$ is the product $G = ZK$ of a central subgroup $Z$ and a non-Abelian subgroup $K$, then $r(Z) \leq r(Z \cap K) + \bar{r}(G)$. If $Z \cap K = 1$, then $r(Z) \leq \bar{r}(G)$.

**Proof.** Since $G/K = Z/Z \cap K$, it follows that $r(Z) \leq r(Z \cap K) + r(G/K)$. Therefore the desired inequality is a consequence of the relation $r(G/K) \leq \bar{r}(G)$ proved in Lemma 1.

**LEMMA 2.** If $G$ is a non-Abelian finite or solvable group, then $r(Z(G)) \leq \bar{r}(G)$.

**Proof.** Consider first the case of a finite group $G$. Choose in $G$ a minimal non-Abelian subgroup $F$ (a Miller-Moreno subgroup). It follows from a description of the structure of finite minimal non-Abelian groups [4, pp. 285, 309] that the rank of the center of $F$ is at most 3. Applying the corollary of Lemma 1 to the subgroup $Z(G)F$, we obtain $r(Z(G)) \leq r(Z(G) \cap F) + \bar{r}(G) \leq 3 + \bar{r}(G)$.

Now suppose the group $G$ is solvable and assume that $r(Z(G)) > 3 + \bar{r}(G)$. Choose in the center $Z(G)$ of $G$ a finitely generated subgroup $Z$ such that

$$r(Z) > 3 + \bar{r}(G),$$

and choose in $G$ a finitely generated, met-Abelian subgroup $H$. The product $H_1 = ZH$ is a finitely generated, met-Abelian subgroup. Suppose $p$ is a prime such that

$$r(Z) = r(Z/Z^p).$$

We use the result on the residual finiteness of finitely generated, met-Abelian groups, which follows from theorems of Hall [5]. In accordance with this result, the subgroup $H_1$ contains a normal subgroup $M$ of finite index such that the factor group $H_1/M$ is non-Abelian and the group $H_1Z$ contains a normal subgroup $N$ of finite index such that $Z \cap N = ZP$.

The factor group $H_1/M \cap N$ is finite and non-Abelian and its center contains the subgroup $Z(M \cap N)/M \cap N$ which is isomorphic, in view of the relation $Z \cap N = ZP$, to the factor $Z/ZP \cap N$. It then follows from the fact on finite groups proved above that the rank of $Z/ZP \cap N$ is at most $3 + \overline{r}(G)$, hence $r(Z/ZP) \leq 3 + \bar{r}(G)$. In view of (2), this contradicts assumption (1) on the rank of $Z$. The lemma is proved.

**LEMMA 3.** The wreath product $W$ of a group of prime order $p$ and an infinite cyclic group has infinite non-Abelian rank.

**Proof.** Suppose $A$ is the base of the wreath product $W$, $W = A\langle g \rangle$, $\langle g \rangle$ is an infinite cyclic group and $V = A\langle g^n \rangle$ is a subgroup of $W$ where $n$ is any natural number. The subgroup $A$ is a direct product $A = A_1 \times \ldots \times A_n$ of $g^n$-admissible subgroups $A_i$, $i = 1, \ldots, n$, such that the product $A_i\langle g^n \rangle$ is isomorphic to $W$. Put $B_i = [A_i, g^n, g^n]$, $B_i = [A_i, g^n]$, $2 \leq i \leq n$. The factor group $V = V/B_1 \times B_2 \times \ldots \times B_n$ has the decomposition $V = A\langle g^n \rangle = (A_1\langle g^n \rangle) \times A_2 \times \ldots \times A_n$, where $A_1\langle g^n \rangle$ is a non-Abelian group, $|A_1| = p$, $2 \leq i \leq n$. Applying the assertion of Lemma 1 to the group $V$ and its subgroup $A_1\langle g^n \rangle$, we obtain $r(V/A_1\langle g^n \rangle) = r(A_2 \times \ldots \times A_n) \leq \bar{r}(V) \leq \bar{r}(W)$. Consequently, $n - 1 \leq r(W)$ and, therefore, since $n$ is arbitrary, the non-Abelian rank $r(W)$ is infinite.

**COROLLARY.** Suppose a group $G$ is a product $G = A\langle g \rangle$, where $A$ is a periodic Abelian normal subgroup. If the non-Abelian rank of $G$ is finite, then $A$ is a union of finite normal subgroups of $G$.

**Proof.** It suffices to consider the case where $A$ is a $p$-group, $p$ a prime, and $g$ is an element of infinite order. We will show that any element $a \in A$ is contained in a finite normal subgroup of $G$. Suppose $p^n$ is the order of $a$ and $A^k = \langle \langle a^{p^k} \rangle \rangle$ is the normal closure of $a^{p^k}$, $k = 0, 1, \ldots, n$ in $G$.

Consider the series of subgroups

$$A_0 > A_1 > \ldots > A_n = 1$$

and put $G = G/A_{k+1}$. The subgroup $A_k = A_k/A_{k+1}$ of $G$ is a factor of the series (3) and is generated by the elements...