LITERATURE CITED


PRECISE ESTIMATE OF THE 2-CAPACITY OF A CONDENSER

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For quite a large class of condensers $E$ (including, in particular, all space annuli) greatest lower bounds for their 2-capacity are obtained in terms of the Newtonian capacity of certain sets associated with $E$. A class of condensers for which equality is achieved in the bound is described completely.

1. A pair $E = (E^+, E^-)$ of nonempty separated sets $E^+$ and $E^-$ in Euclidean space $\mathbb{R}^p$, $p \geq 3$, one of which is bounded, is called a condenser. Let $\text{cap} E$ be the 2-capacity of a condenser $E$ [1], defined as the greatest lower bound of $\int_{\mathbb{R}^p} |\text{grad} f(x)|^2 dx$, where $f$ runs through the set of all real-valued functions in $\mathbb{R}^p$ which are continuous in $\overline{\mathbb{R}^p} := \mathbb{R}^p \cup \{-\infty \}$ and absolutely continuous on linear manifolds in $\mathbb{R}^p$ and which take on the values 1 and 0 on $E^+$ and $E^-$, respectively.

For quite a large class of condensers (including, in particular, all space annuli [1]), we obtain greatest lower bounds for $\text{cap} E$ in terms of the Newtonian capacity of certain sets associated with $E$. We describe completely a class of condensers on which equality in these bounds is achieved.

For a set $Q \subset \mathbb{R}^p$ we let $\partial Q$ and $CQ$ denote the boundary of $Q$ (in $\mathbb{R}^p$) and the complement of $Q$ with respect to $\mathbb{R}^p$, respectively.

2. For definiteness, suppose that $E^+$ is compact. It is natural to assume that $E^+$ has nonzero Newtonian capacity: $C_2(E^+) > 0$, since in the contrary case $\text{cap} E = 0$.

For a closed set $F \subset \mathbb{R}^p$ we let $\mathcal{F}$ denote its reduced kernel [2]. The open set $\mathcal{C}E^-$ consists of a finite or countable number of regions $Z_i$, $i \in I \subset \mathbb{N}$. Let $I_+ = \{i \in I : \text{for which the sets } Z_i \cap E^+ \text{ are nonempty} \}$. We put $Z_+ := \bigcup_{i \in I_+} Z_i$. In what follows we will assume that $CZ_+$ is unbounded unless otherwise stipulated.

It is easy to see (using results from [3], e.g.) that
\[ \text{cap} E \geq (p - 2) \omega_p C_2(E^+) \]
where \( \omega_p \) is \((p - 1)\)-dimensional Lebesgue measure of the sphere \(|x| = 1\). Our goal is to improve the estimate (1) under the assumption that
\[ C_2(Z_+) < \infty \] (2)
In particular, (2) is fulfilled for every annulus, i.e., every condenser \( E \) for which \( E^+ \) and \( E^- \) are the bounded and unbounded components of the complement with respect to \( \mathbb{R}^d \) of some doubly connected bounded region. In the general case, we do not require that the boundary \( \partial Z_+ \) be compact or that the "gap" \( C(E^+ \cup E^-) \) of \( E \) be connected.

Let \( \Omega \) be the unbounded connected component of \( CE^+ \), let \( \gamma_{E^+} \) be the equilibrium measure of \( E^+ \) [2], and let \( u(x) := u_{E^+}(x) \) be its Newtonian potential. We will denote by \( \Gamma \) the collection of equipotential sets \( \{ x \in \Omega : u(x) = c \} \) of functions \( u(x) \) which are harmonic in \( \Omega \). The elements of \( \Gamma \) are closed bounded sets.

**THEOREM 1.** We have that
\[ \text{cap} E \geq \frac{(p - 2) \omega_p C_2(Z_+) C_2(E^+)}{C_2(Z_+) - C_2(E^+)} \] (3)
and the equal sign in (3) holds if and only if \( \partial Z_+ \in \Gamma \).

In particular, for a spherical annulus \( E_0 = (E_0^+, E_0^-) \) with \( E_0^+ = \{ x : |x| < a \} \) and \( E_0^- = \{ x : |x| > b \}, 0 < a < b < \infty \), we get the well-known equality [1]
\[ \text{cap} E_0 = \frac{(p - 2) \omega_p}{a^{p-2} - b^{p-2}} \] (4)

**COROLLARY 1.** Suppose that the condenser \( E \) and the spherical annulus \( E_0 \) are related by the conditions \( C_2(E^+) = C_2(E_0^+) (= a^{p-2}) \) and \( C_2(Z_+) = C_2(\partial E_0^-) (= b^{p-2}) \). Then
\[ \text{cap} E \geq \text{cap} E_0 \] (5)
The inequality (5) follows from (3) and (4). A statement analogous to Corollary 1 is stated in [1], in which the Lebesgue measures of the appropriate sets, instead of their capacities, are assumed to be invariant.

3. We restate (3) in an equivalent form. For this, we will need the following concepts from potential theory [2].

For a closed set \( F \subset \mathbb{R}^d \), we denote by \( \mathcal{M}(F) \) the collection of all unit measures \( \mu \) with support \( S(\mu) \subset F \). If \( C_2(F) \in (0, \infty) \), we put \( W_2(F) := 1/C_2(F) \). (In case \( F \) is compact, \( W_2(F) \) is called the Robin constant of \( F \).) Then
\[ \mathcal{J}_2(\omega_F) = W_2(F) < \mathcal{J}_2(\mu) \text{ } \forall \mu \in \mathcal{M}(F), \mu \neq \omega_F, \] (6)
where \( \omega_F := \gamma_F/C_2(F) (\in \mathcal{M}(F)) \), and \( \mathcal{J}_2(\cdot) \) is the Newtonian energy functional. It is well-known that \( \omega_F \) is a unit measure in the class \( \mathcal{M}(F) \) whose Newtonian potential is constant quasieverywhere on \( F \).

Let \( E = (E^+, E^-) \) be an arbitrary condenser, let \( \mathcal{M}(E) \) be the class of all charges \( \nu = \nu^+ - \nu^- \), for which \( \nu^+ \in \mathcal{M}(E^+), \nu^- \in \mathcal{M}(E^-) \), and let \( V_2(E) \) be the extremal characteristic of \( E \), defined by
\[ V_2(E) = \inf_{\nu \in \mathcal{M}(E)} \mathcal{J}_2(\nu). \]

As is shown in [3], for each \( E \) which satisfies either the condition a) \( E^- \) is compact, or the condition b) \( C_2(CZ_+) = \infty \) (and, in particular, for \( E \) satisfying (2)), we have that
\[ \text{cap} E \geq (p - 2) \omega_p / V_2(E) \] (7)