BOUNDARY PROBLEM WITH DISCONTINUOUS TRANSLATION FOR TWO FUNCTIONS
WHICH ARE ANALYTIC IN DOMAINS OF DIFFERENT CONNECTIVITY

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In the space $L_p(\Gamma)$, $1 < p < \infty$ we consider the problem of finding functions $\Psi$ and $\psi$ which are analytic, respectively, in a doubly connected domain with boundary $\Gamma$ and a simply connected domain with boundary $\gamma$ ($\Gamma$ and $\gamma$ are closed Lyapunov contours) with respect to the boundary condition $a(t)\psi'[\alpha(t)] + b(t)\psi[\alpha(t)] + c(t)\psi(t) + d(t)\overline{\psi(t)} = h(t)$, $t \in \Gamma$, where the orientation-preserving translation $\alpha: \Gamma \to \gamma$ and the coefficients $a$, $b$, $c$, $d$ admit discontinuities at certain points of $\Gamma$. Under specific conditions on the limit values of the derivative of the translation at the points of discontinuity we get necessary and sufficient conditions for the problem to be Noetherian and a formula for calculating its index.

Let $\Gamma_1$, $\Gamma_2$, and $\gamma$ be simple closed pairwise disjoint Lyapunov curves. The contour $\Gamma = \Gamma_1 \cup \Gamma_2(\gamma)$ bounds a finite doubly connected (simply connected) domain $D(\delta)$. As positive circuit of the boundary $\Gamma(\gamma)$ we take the one for which the domain $D(\delta)$ remains on the left. By $[\tau, t]$ we denote the closed arc of the curve $\Gamma$ or $\gamma$ run through from the point $\tau$ to the point $t$ in the positive direction; $j = 1, n$ will mean that $j = 1, \ldots, n$. On the contour $\Gamma$ points $\lambda_1 \in \Gamma_1$, and $\lambda_2 \in \Gamma_2$ are chosen and on $\gamma$ points $\lambda_3$ and $\lambda_4$. Let $\alpha(\lambda)$ be the set of all values of a given piecewise-smooth translation $\alpha: \Gamma \to \gamma$ at the point $\lambda \in \Gamma$ admitting discontinuities at the points $\lambda_1$ and $\lambda_2$, preserving orientation, and mapping the set $\Gamma' = \Gamma\{\lambda_1, \lambda_2\}$ homeomorphically onto the set $\gamma\{\lambda_3, \lambda_4\}$ so that the arcs $\Gamma_1$ and $\Gamma_2$ are carried by the translation onto the arcs $[\lambda_3, \lambda_4]$ and $[\lambda_4, \lambda_3]$, respectively, where $\alpha(\lambda_1) = \alpha(\lambda_2) = \{\lambda_3, \lambda_4\}$. We also assume that $\inf_{t \in \Gamma} |a'(t)| > 0$. Let $F = \bigcup_{j=1}^{n} \lambda_j$ be the set of nodes of the contour $\mathcal{J} = \Gamma \cup \gamma$. We consider the boundary problem

$$a(t)\psi'[\alpha(t)] + b(t)\psi[\alpha(t)] + c(t)\psi(t) + d(t)\overline{\psi(t)} = h(t), t \in \Gamma,$$

where $h \in L_p(\Gamma)$, $1 < p < \infty$, $\Psi$ and $\psi$ are functions which are analytic, respectively, in $D$ and $\delta$, $a$, $b$, $c$, and $d$ are piecewise-continuous functions on $\Gamma$ admitting discontinuities at the nodes. With the help of the operator approach [1, 2] we establish a criterion for the problem (1) to be Noetherian and calculate its index.

We introduce operators: $(S_L\psi)(t) = (\pi t)^{-1}\int_r^\tau (\tau - t)^{-1}\psi(\tau)\,d\tau$, $(L_\psi)(t) = \psi(t)$, $t \in L$; $P_L = (I_L + S_L)/2$, $Q_L = I_L - P_L$; $(B_\alpha\psi)(t) = \psi[\alpha(t)]$; $(C_\psi)(t) = \overline{\psi(t)}$. Let $|\alpha|$ be the restriction of a map, $L_p^+$

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(L) = Im \mathcal{P}_L | LP(L), \ \tilde{p}_p^{-1}(L) = \text{Im} \mathcal{Q}_L | LP(L), \ T \text{ be the transpose, } \ast \text{ be equality up to a completely continuous operator, } \ast \text{ be the direct sum either of spaces or operators; } A^{-1} \text{ be the regularizer of the Noetherian operator } A, \text{ ind } A \text{ be its index; } \mathcal{L}(X,Y) \text{ be the algebra of bounded linear operators acting from the Banach space } X \text{ to the Banach space } Y, \mathcal{L}(X, X); \text{ we call the operators } A \text{ and } B \text{ equivalent if they are simultaneously Noetherian and } \text{ind } A = \text{ind } B. \text{ That (1) is Noetherian is equivalent to the } (1 \times 2)-\text{matrix}

\begin{align*}
T_0 &= \begin{pmatrix}
(a + b\mathcal{C})B, & c + d\mathcal{C}
\end{pmatrix} \in \mathcal{L}(L^+_{\mathcal{G}}(\gamma) \oplus L^+_{\mathcal{G}}(\Gamma), L^+_{\mathcal{G}}(\Gamma))
\end{align*}

being Noetherian, and the index of the problem (1) is equal to \text{ind } T_0. \text{ As a preliminary we study the question of when the operator } \Pi = (P_B B^{-1}, P_B \mathcal{C})^T \in \mathcal{L}(LP(\mathcal{G}), LP(\mathcal{G}) \oplus LP(\mathcal{G})) \text{ is Noetherian. We introduce operators: } \Pi_1 = (P_B, \mathcal{C}) \in \mathcal{L}(LP(\mathcal{G}) \oplus LP(\mathcal{G})) \text{; } \Pi_2 = I_{\mathcal{G}} + I_{\mathcal{G}} \in \mathcal{L}(LP(\Gamma)), \text{ where } H_B = B_B P_B B^{-1} - P_B (H_B \neq 0 \text{ by virtue of the discontinuity of the translation } \alpha) ; K_1 = \begin{pmatrix}
P_B & P_B B^{-1}
\end{pmatrix} \in \mathcal{L}(LP(\gamma) \oplus LP(\Gamma)); K_2 = \begin{pmatrix}
P_B & P_B B^{-1}
\end{pmatrix} \in \mathcal{L}(LP(\Gamma) \oplus \mathcal{G}^{-1}(\gamma)). \text{ We note that in reference to [3] and [4] we will have in mind the familiar matrix equality in [3, p. 4] and Lemma 3 in [4, p. 1143], respectively.}

**LEMMA 1.** The operator } \Pi \text{ admits left (right) regularization if the operator } \Pi_1 \text{ (the operators } K_1 \text{ and } K_2 \text{) is (are) Noetherian.}

**Proof.** Left regularization follows from the relation } \Pi_1 \Pi = \Pi_2. \text{ According to [3], the operator } \Pi_1 = \begin{pmatrix}
P_B & P_B B^{-1}
\end{pmatrix} \in \mathcal{L}(LP(\gamma) \oplus LP(\Gamma)) \text{ is equivalent to the operator } U^T X_1 = K_1 \text{ where } U \text{ is a nondegenerate matrix, from which right regularization follows.}

The lemma is proved.

We introduce the Carleman translation } \beta: \mathcal{J} \rightarrow \mathcal{J} \text{ of the second order such that } \beta(t) = \alpha(t), \text{ if } t \in \Gamma, \text{ and } \beta(t) = \alpha(t), \text{ if } t \in \gamma. \text{ We note that } \beta \text{ has discontinuities on } \mathcal{J}. \text{ Let } X_B \text{ be the operator of multiplication by the characteristic function of the set } L \subset \mathcal{J}: X_B = B_B P_B B^{-1} - P_B, \text{ where } (B_B P_B B^{-1} - P_B) = \phi (t).

**LEMMA 2.** The operators } \Pi_2, K_1 \text{ and } K_2 \text{ are Noetherian if and only if the operators } \Pi_3 = I_{\mathcal{J}} X_B H_B \in \mathcal{L}(\mathcal{J}),

\begin{align*}
K_3 &= \begin{pmatrix}
I_{\mathcal{J}} & X_B P_B B^{-1}
\end{pmatrix} \in \mathcal{L}(L^2(\mathcal{J})) \text{ are, respectively, Noetherian.}
\end{align*}

**Proof.** We establish, e.g., the equivalence of the operators } K_1 \text{ and } K_3 \text{ (the other assertions of the lemma are proved by following the same scheme). The operators } K = \begin{pmatrix}
P_B & B^{-1}
\end{pmatrix} \text{ and } K_3 = \begin{pmatrix}
P_B & B^{-1}
\end{pmatrix} \text{ act on the space } LP(\gamma) \oplus LP(\Gamma). \text{ Since } \Pi^+ = \text{diag } \{P_B, Q_B\} \text{ and } \Pi^- = \text{diag } \{Q_B, P_B\} \text{ are complementary projectors of the space } LP(\gamma) \oplus LP(\Gamma) \text{ onto the subspaces } LP(\gamma) \oplus LP(\Gamma) \text{ and } LP(\gamma) \oplus LP(\Gamma), \text{ respectively, and } K_1 = \Pi^+ K_{11} + K_5 = \Pi^- \Pi^+, \text{ one has [4] that } K_1 \text{ and } K_2 \text{ are equivalent. Further, let } R^+ = \text{diag } \{X_B, X_B\} \text{ and } R^- = \text{diag } \{X_B, X_B\} \text{ be complementary projectors of } LP(\mathcal{J}) \text{ onto the subspaces } LP(\mathcal{J}) \oplus LP(\mathcal{J}) \text{ and } LP(\mathcal{J}) \oplus LP(\mathcal{J}); \text{ respectively, and let } K_5 = \begin{pmatrix}
I_{\mathcal{J}} & P_B B^{-1}
\end{pmatrix} \in \mathcal{L}(L^2(\mathcal{J})). \text{ Since } K_5 = R^+ K_5 R^+ \text{ and } K_5 = R^+ K_5 R^+, \text{ one has [4] that } K^+ \text{ and } K_3 \text{ are equivalent. The equivalence of } K_1 \text{ and } K_3 \text{ follows from this. The lemma is proved.}

Let } \gamma_1 S \text{ and } \gamma_2 S \text{ be arcs from a neighborhood on } \mathcal{J} \text{ of the node } \lambda_0 \in \mathcal{J}, \ s = 1, 4 \text{ such that } \gamma_1 S \text{ issues from } \lambda_0 \text{ and } \gamma_2 S \text{ ends at } \lambda_0, \text{ while } \gamma_1 S \cap \gamma_2 S = \emptyset (s \neq r; s, r = 1, 4; j, k = 1, 2); \text{ } a_j = \lim_{t \to \gamma_1} a_j(t) (\text{here } t \in \gamma_1 \cap \Gamma), j = 1, 2; \nu = e^{z}_2 / (e^{z}_2 - 1), \text{ where } z = \pi(x + ip^{-1}), x \in \Gamma = (-\infty, \infty); W_{J_0} = \left| (b^s) J_0 \right|^0 [x - 1/p_0, s = 1, 4; j = 1, 2; \delta = W_1 W_2, \text{ where } W_1 = W_{11} / W_{22}, W_2 = W_{21} / W_{12}. \text{ The next lemma follows from Theorem 4 in [5].}