Mean-Value Theorems and the Nonoscillatory Nature of Solutions of Partial Differential Equations in the Spaces $\mathbb{E}^n$ and $\mathbb{P}^n$

M. K. Bugir

Analogs of the mean-value theorem are obtained for the Poisson equation with the Laplace-Beltrami operator in Euclidean and spherical spaces.

1. The mean-value theorem plays an important role in the study of the properties of harmonic functions (see [1], for example).

**Theorem 1.** The regular function $u(x)$ is harmonic in a region $D$ if and only if the relation

$$
\frac{1}{\omega_n r^{n-1}} \int_{S_r} u(x) ds = u(P_0)
$$

(1)

is satisfied for every sphere $S_r$ in $D$ with center at $P_0$.

The $\omega_n$ in (1) denotes the area of the $n$-dimensional unit sphere.

In the present paper, Theorem 1 is generalized to a partial differential equation of more general form.

For simplicity, we consider first the equation

$$
Lu + pu = f(x)
$$

(2)

with the Laplace-Beltrami operator $L$, which in Euclidean space $\mathbb{E}^n$ is the Laplace operator and in hyperbolic space the d'Alembert operator, as its principal part; thus, it is possible to study equations of different types by a single approach.

We denote by $X$ a space of constant curvature with metric

$$
dx = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j,
$$

where $\{g_{ij}\}$ is the metric tensor. We introduce the "mean value"

$$
M_r[u(x), P_0] = \frac{1}{A(r)} \int_{S_r} u(x) ds,
$$

(3)

where $A(r)$ denotes a normalizing coefficient such that $M_r(1, P_0) = 1$, and $S_r$ is the sphere of radius $r$ with center at $P_0 \in X$.

If we integrate (2) over the sphere $S_r$ in $X$ and use the formula [2], [3]

$$
L_r M_r = M_r L_r,
$$

(4)

where $L_r$ denotes the radial part of $L$ in spherical polar-geodesic or polyspherical coordinates, depending on the type of space, we get

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\[
\frac{d}{dr} \left[ A(r) \frac{dM(r)}{dr} \right] + pA(r)M(r) = \frac{1}{\omega_n} \int_{s}^{r} f(x) dx.
\]

We let

\[\int_{s}^{r} f(x) dx \cdot ds = F(r).\]

Depending on the type of the space \(X\), (5) becomes Bessel's or Legendre's equation. We consider the Euclidean space \(E^n\), for which (5) has the form

\[
\frac{d}{dr} \left[ r^{n-1} \frac{dM(r)}{dr} \right] + p r^{n-1} M = \frac{1}{\omega_n} \int_{s}^{r} f(x) ds.
\]

(5')

If we put \(M(r) = r^{-(n-2)/2}\psi(r)\), \(\tau = \sqrt{p sgn p} r\), then (5) becomes the Bessel equation

\[
\frac{d^2 \psi}{dt^2} + \frac{1}{t} \frac{d \psi}{dt} + \left( \pm 1 - \frac{\nu^2}{t^2} \right) \frac{1}{p \omega_n} \left( \frac{\tau}{\sqrt{p}} \right)^{-\nu+1} F \left( \frac{\tau}{\sqrt{p}} \right),
\]

where we take the "+" sign for \(p > 0\) and the "-" sign for \(p < 0\), and \(\nu = (n - 2)/2\). The general solution of (6) can be written in the following way by means of Bessel functions:

\[
\psi(r) = J_\nu(\tau)C_1 + N_\nu(\tau)C_2 + \int_{0}^{\tau} \left( \frac{\pi}{2} \right) J_\nu(s)N_\nu(\tau) -
\]

\[-N_\nu(s)J_\nu(\tau) \left( \frac{s}{\sqrt{p}} \right)^{-\nu/2} F \left( \frac{s}{\sqrt{p}} \right) ds \frac{1}{p \omega_n}.\]

(7)

In case \(p < 0\), we must use \(I_\nu(t)\) instead of \(J_\nu(t)\). We let \(K(t, r)\) denote the Cauchy function of the homogeneous equation corresponding to (6); then

\[
M(r) = r^{-(n-2)/2} \left[ J_\nu(\sqrt{p} r)C_1 + N_\nu(\sqrt{p} r)C_2 \right] + \frac{1}{\omega_n p} \int_{0}^{\sqrt{p} r} K(r, s)s^{-\nu/2} F(s) ds.
\]

(8)

Using the properties of the Bessel function, we have

\[
\lim_{r \to 0} r^{-(n-2)/2} J_\nu(\sqrt{p} r) = \frac{1}{2\nu \Gamma(\nu + 1)}; \quad \lim_{r \to 0} r^{-(n-2)/2} N_\nu(\sqrt{p} r) = \infty;
\]

therefore if we are interested in regular solutions of (5), we must set the constant \(c_2\) in the general solution (8) equal to zero. Since \(\lim_{r \to 0} M(r) = u(P_0)\), normalizing the fundamental solution \(r^{-\nu} J_\nu(\sqrt{p} r)\) by means of the conditions \(M_1(0) = 1\) and \(M_1'(0) = 0\), we get a regular solution of (5) in the form

\[
M(r) = \frac{\Gamma(\nu/2)J_{(n-2)/2}(\sqrt{p} r)}{(\sqrt{p} r)^{(n-2)/2}} u(P_0) + \frac{1}{\omega_n p} \int_{0}^{\sqrt{p} r} s^{-\nu/2} K(r, s) F(s) ds.
\]

(9)

2. We must now establish when the converse of the above statement concerning (9) holds, i.e., given a regular solution in the form (9), to reconstruct the form of an equation which it satisfies, as was done with the Laplace equation.