A THEOREM ON CONTINGENTS OF HYPERSPACES IN EUCLIDEAN SPACE

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An attempt is made to carry out a deeper study of the contingent characterization of arbitrary sets in euclidean space $\mathbb{R}^{m+1}$.

Let $E$ be an arbitrary set in euclidean $(m + 1)$-space $\mathbb{R}^{m+1}$. A ray $\ell$ beginning at a point $A(x) \in E \subset \mathbb{R}^{m+1}$, where $x = (x_1, x_2, ..., x_{m+1})$ (where $A$ is a nonisolated point of $E$) is called an intermediate half-tangent of $E$ at $A$ if there exists a sequence of points $\{A_n\} \subset E$, converging to $A$, such that the sequence of rays $\{AA_n\}$ converges to $\ell$.

The set of all intermediate half-tangents of $E$ at $A$ is called the contingent of $E$ at $A$, denoted by $\text{cont}_{E}A$ [1].

Let $A \in E$ be a limit point of $E$. Take the unit sphere $S^m(A)$ at that point. The intersection of $\text{cont}_{E}A$ with the unit sphere $S^m(A)$ is known as the spherical contingent, denoted by

$$\text{cont}_{E}^S A = \text{cont}_{E}A \cap S^m(A).$$

Let $V_\varepsilon\cdot(A)$ be the $\varepsilon$-neighborhood of $A$, where $\varepsilon > 0$. Project all the points of $E \cap V_\varepsilon\cdot(A)\{A\}$ by rays beginning at $A$ onto $S^m(A)$. Denote the resulting set by $M_\varepsilon(A)$.

It can be proved that the intersection of the closures of $M_\varepsilon(A)$ for all values of $\varepsilon > 0$ is the spherical contingent:

$$\text{cont}_{E}^S A = \text{cont}_{E}A \cap S^m(A) = \bigcap_{\varepsilon \to 0} M_\varepsilon(A).$$

The question of the contingent structure of an arbitrary set in $\mathbb{R}^{m+1}$ in the measure sense was solved in [1].

As to the contingent characterization of sets of second category, there is no definitive solution to the problem even for the graph of a continuous function $u = f(x)$, where $x = (x_1, x_2, ..., x_m)$, though some results are nevertheless available [2]. It has been proved that for any locally compact set $E \subset \mathbb{R}^{m+1}$ the contingent $\text{cont}_{E}^S A$ at points of a subset of second category is a centrally symmetric closed set of rays. This paper is devoted to further refinement of the contingent characterization for an arbitrary set, in the case of the graph $\Gamma$ of a continuous function $u = f(x)$, where $x = (x_1, x_2, ..., x_m)$ in some domain $D \subset \mathbb{R}^m$. Given the graph $\Gamma$ of a single-valued function $u = f(x)$, we now define the cylindrical con-
tingent of the set at an arbitrary point \( A_0 \in \Gamma \). Let \( S^{m-1}(x_0) \) be the boundary of the unit ball \( V_m(x_0) \subset \mathbb{R}^m \), \( x_0 = \text{pr}_E \in D \); we will denote points on \( S^{m-1}(x_0) \) by \( \phi \), sometimes calling them directions at \( x_0 \). Compactify the cylinder \( C^m(A_0) = S^{m-1}(x_0) \times \mathbb{R}^1(u) \) with generators \( \mathbb{R}^1(u) \) parallel to the axis \( Ou \), by adding to each generator \( \mathbb{R}^1(u) \) two points at infinity, \( \pm \infty \); denote the compactified cylinder thus obtained by \( \hat{C}^m(A_0) = S^{m-1}(x_0) \times \hat{\mathbb{R}}^1(u) \). A point \( \tilde{A}_0 = (\phi_0, u) \) of \( C^m(A_0) \) will belong to the cylindrical contingent of \( \Gamma \) if there is a sequence of points \( A_k \in \Gamma \) such that:

1) the sequence of rays \( \tilde{A}_0A_k \) converges to some ray \( \tilde{\ell} \in \mathbb{R}^{m+1} \) such that a) \( \tilde{\ell} \) is not parallel to \( Ou \); then the intersection \( \tilde{\ell} \cap C^m \) over the ray \( \Phi_0 \) is denoted by \( u \); b) if \( \tilde{\ell} \) is parallel to \( Ou \), then \( u \) are the ideal points \( \pm \infty \) of \( C^m \) on the ray \( \Phi_0 \).

2) the ray \( \Phi = \Phi_0 \) is a half-tangent for the sequence of points \( \text{pr}_{V_m}A_k \).

The set of all such points is known as the cylindrical contingent of the graph \( \Gamma \) at \( A_0 \), denoted by \( \text{cont}^g_{C^m} \).

It should be noted that the cylindrical contingent is defined only for a single-valued function \( f \).

Consider an arbitrary point \( A_0(x_0, u_0), x_0 \in D \), of the graph \( \Gamma \) of a continuous function \( f \). Denote the points of this graph over a deleted \( \delta_q \)-neighborhood \( U_q(x_0) \setminus \{x_0\} \subset D \) by \( \Gamma_q \); clearly, \( \Gamma_q \) is connected. Denote the projection of \( \Gamma_q \) from \( A_0 \) onto \( C^m(A_0) \) by \( M_q \); \( M_q \) is also connected, and moreover \( M_q \supset M_{q+1} \). It is easy to see that

\[
\text{cont}^g_{C^m} A_0 = \bigcap_q M_q .
\]

Hence it follows that \( \text{cont}^g_{C^m} A_0 \subset C^m \) is a continuum. We will prove that the intersection of this continuum with an arbitrary generator \( \Phi = \Phi_0 \) is also connected. Indeed, take a ray \( \tilde{\ell} \subset D \) beginning at a point \( x_0 \in D \) which intersects the sphere \( S^{m-1}(x_0) \) at a point \( \Phi_0 \) and consider the conical neighborhood \( \Omega_q \subset D \) of the ray, defined as the set of all rays that form an angle less than \( \delta_q \) with \( \tilde{\ell} \); we will again assume that \( \delta_q \downarrow 0 \). Denote the points of the graph \( \Gamma \setminus \{A_0\} \) over \( \Omega_q \) that belong to the \( 1/q \)-neighborhood of \( A_0 \) by \( \Gamma_q(\Phi_0) \). Obviously, \( \Gamma_q(\Phi_0) \) is connected. Denote the projection of \( \Gamma_q(\Phi_0) \) from \( A_0 \) onto \( C^m \) by \( M_q(\Phi_0) \); \( M_q(\Phi_0) \) is also connected, and moreover \( M_q(\Phi_0) \supset M_{q+1}(\Phi_0) \). It is easy to show that the intersection of \( \text{cont}^g_{C^m} A_0 \) with a generator \( (\Phi_0, \mathbb{R}^1) \) of \( C^m \) is \( \bigcap_q M_q(\Phi_0) \). It follows that this intersection is connected.

Let us call a number \( a \) (possibly \( a = \frac{\partial f}{\partial \Phi} \)) a derivate of a function \( f \) in the direction \( \Phi \) if there is a sequence of points \( B_k(x_k) \in D \), \( k = 1, 2, \ldots \), that converges to \( B_0 \) in such a way that the sequence of directions of the vectors \( \overrightarrow{B_0B_k} \) converges in direction to \( \Phi \), and moreover

\[
(f(B_k) - f(B_0)) \big/ |B_0B_k| \to a.
\]

We will consider the cylinder \( C^m(A_0) \) relative to a cartesian coordinate system \( \{X, U\} \) translated parallel to itself to the point \( A_0 \) of the graph \( \Gamma \) of \( f \): \( X = x - x_0 \), \( U = u - u_0 \). Then it will not be hard to show that if \( a \) is a derivate of \( f \) in the direction \( \Phi \), then the point with coordinates \( (\Phi, a) \) belongs to \( \text{cont}^g_{C^m} A_0 \), and conversely: every point of the contingent is a derivate of \( f \) in a suitable direction.

**Definition.** A continuum \( K \) on the cylinder \( C^m \) (and on \( S^m \) if \( K \) does not contain the points at infinity \( \pm \infty \)) is said to be regular if every generator of the cylinder (semi-meridian of the sphere) intersects it in a connected set.

If \( K \subset C^m \) is a regular continuum, there arise two naturally defined functions \( P(\Phi) \) and \( Q(\Phi) \) on \( S^{m-1} \): we let \( (\Phi, P(\Phi)) \) and \( (\Phi, Q(\Phi)) \) be the points of \( K \) which are, respectively, the lowest and uppermost points of \( K \) on the generator \( \Phi = \text{const} \) of \( C^m \). It is easy to see that \( P(\Phi) \) and \( Q(\Phi) \) are connected functions of \( \Phi \) on \( S^{m-1} \), and are constant on each connected component of \( S^{m-1} \) which contains \( K \).