SOLITARY WAVES IN A LAYER OF VISCOUS LIQUID

V. Ya. Shkadov

Solitary waves in a thin layer of viscous liquid which is running down a vertical surface under the action of gravity are investigated. The existence of such waves was demonstrated in the experiments of [1, 2]. The difficulties that must be faced in a theoretical computation were also noted in these studies. Below a solution of the problem of stationary waves is obtained by the method of expansion in the small parameter in two regions with subsequent matching and also by a numerical integration method. It is shown that in each case a solution of solitary wave type exists along with the single-parameter family of periodic solutions (parameter - the wave number \( \alpha \)). On decreasing the wave number, the periodic waves go over into a succession of solitary waves.

As the basis of the investigation we take the equation for the thickness of the layer \( h(\xi) \), which is obtained by integrating the basic equations of motion of a viscous liquid transverse to the layer. In the integration it is assumed that the boundary-layer approximation can be used and a parabolic profile of the longitudinal velocity is taken. In the coordinate system attached to the wave this equation has the following form:

\[
G h'''' + \frac{1}{5} \left[ \theta \left( \frac{1-z}{h} \right)^2 - z^2 \right] h' + H h - E \frac{1+zh-z}{h^3} = 0
\]

Here \( c \) is the wave velocity; \( U_0 \) and \( a_0 \) are the characteristic values of the velocity and the thickness of the layer. For a fixed number of terms considered in this solution, the accuracy decreases with the decreasing wave number \( \alpha \) due to the fact that for small values of \( \alpha \) the wave profile is very different from a harmonic wave. As an example, some results of direct numerical integration of Eq. (1) in the case of periodic waves in a layer of water are shown in Fig. 1 for \( Re = 3 \xi_0 U_0 \nu^{-1} = 24.41 \) and \( \alpha = 0.107 \) (curve 1) and \( \alpha = 0.051 \) (curve 3). The integration was done over a wavelength \( \xi_{10} \leq \xi_1 \leq \xi_{10} + 2\pi a^{-1} \). The initial point \( \xi_{10} \) was chosen at the crest of the wave \( h'(\xi_{10}) = 0 \); for given values of \( Re \) and \( \alpha \), the values of \( c \), \( a_0 \), and the initial data \( h(\xi_{10}) \), \( h''(\xi_{10}) \) were chosen in such a way that for \( \xi_1 = \xi_{10} + 2\pi a^{-1} \) the periodicity condition is satisfied. For the values of \( \alpha \) lying close to the neutral stability curve in the \( Re, \alpha \) plane the wave profiles are almost sinusoidal. The effect of the nonlinear terms in Eq. (1) increases with the decrease of \( \alpha \), and the profiles become noticeably deformed, acquiring the form of solitary waves.

The computations were carried out with a small step along parameter \( \alpha \). For obtaining the wave for \( \alpha_1 = \alpha + \Delta \alpha \) the characteristics of the wave solution corresponding to \( \alpha \) were
used as the initial conditions. For a given value of Re such computations can be carried out only up to a certain finite value $a_k(Re)$. For $a < a_k$ the iteration process of selecting the initial data at the point $\xi_0$ begins to diverge. Thus, although in these computations the tendency for the transition of the periodic waves into solitary waves is detected with decrease of $a$, a special method of solution is needed for determining the solitary waves.

We assume that there is a solitary wave such that for $\xi_1 - \xi_0 + \pm \infty$, $\h - 1$, where $\xi_0$ is some characteristic point on the wave. Let us introduce the change of variable $\xi = Re^{\gamma - 1/2}(\xi_1 - \xi_0)$; then Eq. (1) becomes

$$h'^2 + \delta \omega(z - 1) h' + 6z h - 1 - z(h - 1) = 0$$

$$\delta = 45 - Re^{\gamma - 1}, \omega = 5\zeta - 12\zeta, \zeta = \delta - \delta' = \delta - \delta'$$

We investigate the asymptotic behavior of the small deviation from $h = 1$ for $\xi \to \pm \infty$. We put $h - 1 = \epsilon \exp \sigma \xi$. Linearizing Eq. (2) with respect to $\epsilon$, for $\sigma$ we obtain the following equation:

$$\sigma^2 + \delta \omega \sigma + 3 - \zeta = 0$$

For $\zeta < 3$, Eq. (3) has one real root $\sigma < 0$ and two complex-conjugate roots with positive real part $m \pm i\ell$, $m = -3/2\mu$. Accordingly, we can construct two particular solutions of the linearized equation (2):

$$\psi = \epsilon \exp m\xi, \quad \psi = \epsilon \exp m\xi \cos (b + \ell \xi)$$

We can take $\Phi = 1$ without any loss of generality, since this can always be achieved by a choice of $\epsilon_1$. The decrease of $\epsilon_1$, in turn, is equivalent to displacing the origin for $\xi$ in $\exp \mu \xi$ to the right.

For $\psi$ we write the expansion

$$\psi = \epsilon \psi_1 + \epsilon_1 \psi_2 + \epsilon_2 \psi_3 + ...$$

Substituting (5) into Eq. (2) and equating the coefficients, we obtain

$$\xi_2'' + \delta \omega \xi_2' + (3 - \zeta) \xi_2 = F_k$$

where the right-hand side $F_k$ is expressed in terms of functions $\psi_m$ with smaller numbers and their derivatives. Putting $\rho = \delta \zeta$ and $\Phi = \Phi_k \exp k\mu \xi$, we obtain

$$F_k = (k^2\mu^2 + k^2\omega\mu + 3 - \zeta) F_k \quad (k = 2, 3, ...)
$$

$$F_2 = 2\rho \mu - 3 - 3\mu^2
$$

$$F_3 = (6\rho \mu - 27\mu^2) F_3 + (6\rho \mu - 3 - 3\mu^2) F_2 - (4\rho \mu - 30\mu) F_1 - (3 + 24\mu^2) F_2 - \mu F_1$$

We can take $\Phi_1 = 1$ without any loss of generality, since this can always be achieved by a choice of $\epsilon_1$. The decrease of $\epsilon_1$, in turn, is equivalent to displacing the origin for $\xi$ in $\exp \mu \xi$ to the right.

For $\psi$ we write the expansion

$$\psi = \epsilon \psi_1 + \epsilon_1 \psi_2 + \epsilon_2 \psi_3 + ...$$

We introduce the notation $\xi = b + \ell \xi$; then the coefficients $\psi_k$ in the expansion of $\psi$ in powers of $\epsilon_2$ must be of the form

$$\psi_1 = \exp m\xi \cos \xi, \quad \psi_2 = \exp 2m\xi (\psi_{11} + \psi_{21} \cos 2\xi + \psi_{31} \sin 2\xi)
$$

$$\psi_3 = \exp 3m\xi (\psi_{11} \cos \xi + \psi_{12} \sin \xi + \psi_{31} \cos 3\xi + \psi_{32} \sin 3\xi)$$

We shall restrict the computation to three terms of the expansion (8). Substituting (8) into (2) with (9) taken into consideration, collecting together the terms with equal powers of $\epsilon$ and then the terms with equal harmonics, and equating them to zero, we obtain the equations for $\psi_{k1}$. In particular, for $\psi_{20}$ we have

$$(8m^2 + 28 \delta \omega m + 3 - \zeta) \psi_{20} = -1/2 [3 - 2pm + 3m(m^2 - 3\ell^2)]$$