THE ACTIONS OF INFINITE-DIMENSIONAL LIE ALGEBRAS

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The actions of Lie algebras on smooth manifolds are the subject of a classical and far advanced theory. It is not surprising that the main concepts of this theory may be extended to the infinite-dimensional case; however, in the latter, the theory acquires in an unexpected way what appears to us to be a new and very interesting content. We discuss this infinite-dimensional theory here, and at the end we indicate the connection with problems of calculating the cohomologies of infinite-dimensional Lie algebras.

§1. Definition of the Action. Principal Homogeneous Spaces

1. First we give the main definitions in the most general form. Let $A$ be a nuclear commutative associative $C$-algebra with a unit element, and let $S$ denote the set of closed maximal ideals of $A$, that is, the set of continuous ring homomorphisms $A \to C$. By a tangent vector to $S$ at the point $s \in S$ we mean a continuous linear mapping $d: A \to C$, such that $d(a_1 a_2) = s(a_1) d(a_2) + s(a_2) d(a_1)$. The set $\tau_S$ of these tangent vectors is, in an obvious sense, a linear topological space over $C$; it is called the tangent space to $S$ at $s$. By a vector field on $S$ we mean a derivation of $A$, that is, a continuous linear mapping $D: A \to A$, such that $D(a_1 a_2) = a_1 D(a_2) + a_2 D(a_1)$. The set $\mathfrak{X}(S)$ of these vector fields is, in an obvious sense, a topological Lie algebra and a topological $A$-module. For every $s \in S$ the composition $D \mapsto s \circ D$ defines a homomorphism of the $A$-module $\mathfrak{X}(S)$ into the $C$-module $\tau_S$, compatible with the homomorphism $s: A \to C$. The image of a vector field under this homomorphism $v(s): \mathfrak{X}(S) \to \tau_S$ is called the value of the field at the point $s \in S$.

Let $g$ be a nuclear Lie algebra. We say that $g$ acts on $S$, or that $S$ is a $g$-space, if there is a continuous homomorphism of $g$ into the Lie algebra $\mathfrak{X}(S)$. If the action $\phi: g \to \mathfrak{X}(S)$ has the property that the mapping of the $A$-module $A \otimes g$ into the $A$-module $\mathfrak{X}(S)$ defined by $a \otimes g \mapsto \phi(g)$ is an isomorphism, then $S$ is called a principal homogeneous $g$-space.

2. Next we consider the significance of these definitions in a geometrical situation. The simplest geometrical situation is the case when $S$ is a smooth manifold (finite- or infinite-dimensional) and $A$ is the algebra of complex-valued smooth functions on $S$. In this case the tangent vectors, the vector fields, and the action of the Lie algebra obviously have the usual meaning, and we need interpret only the concept of a principal homogeneous $g$-space.

**Proposition 1.1.** Let $\phi: g \to \mathfrak{X}(S)$ denote the action of the Lie algebra $g$ on the smooth manifold $S$. For $S$ to be a principal homogeneous space relative to this action, it is necessary and sufficient that, for each $s \in S$ the through homomorphism

$$g \ni g \mapsto \phi(s) \ni \tau_S$$

is an isomorphism.

We do not use the necessity of the condition in what follows; therefore we leave the proof of it to the reader; we prove the sufficiency of the condition.

We must prove that the formula \( a \otimes g \mapsto a \varphi (g) \) determines an isomorphism \( A \otimes \mathfrak{g} \rightarrow \mathfrak{g} (S) \). It is obvious that this is a monomorphism, since if \( \sum a_i \otimes g_i \mapsto 0 \), then for every \( s \in S \), under the homomorphism \( v (s) \varphi \), the element \( \sum s (a_i) g_i \in \mathfrak{g} \) goes into the zero element, and this contradicts our assumption. To prove that the mapping is an epimorphism it is sufficient to note that \( A \otimes \mathfrak{g} \) is the space of smooth functions on \( S \) with values in \( \mathfrak{g} \). The element of \( A \otimes \mathfrak{g} \) that goes into a given element \( a \in \mathfrak{g} (S) \) is the function \( s \mapsto (v \circ v (s)^{-1} v (s)) \).

Comment. Proposition 1.1 reveals the extreme poorness of the contact of a principal homogeneous space in the finite-dimensional case. For, if \( \mathfrak{g} \) is the Lie algebra of a finite-dimensional simply connected Lie group \( G \), then the connected components of the principal homogeneous \( \mathfrak{g} \)-space \( S \) must be factor spaces of \( G \) with respect to discrete subgroups. On the contrary, in the infinite-dimensional case this concept is meaningful, as is confirmed by the following example.

3. A basic example. Let \( M \) be a smooth \( n \)-dimensional manifold. Let \( S(M) \) denote the space of \( \infty \)-jets of the mappings \( \mathbb{R}^n \rightarrow M \) which are regular at the point \( 0 \in \mathbb{R}^n \); more precisely, \( S(M) \) is the inverse limit of the sequence

\[
M = S_0 (M) \twoheadrightarrow S_1 (M) \twoheadrightarrow S_2 (M) \twoheadrightarrow \ldots
\]

where \( S_k (M) \) denotes the space of \( k \)-jets of the mappings \( \mathbb{R}^n \rightarrow M \) which are regular at \( 0 \in \mathbb{R}^n \), and the arrows denote the obvious submersion. It is clear that \( S(M) \) is an infinite-dimensional smooth manifold, and that the algebra \( A \) of smooth complex-valued functions on \( S(M) \) is the direct limit of the sequence of algebras \( A_k \) of smooth functions on \( S_k (M) \).

We make the Lie algebra \( W_n \) of formal vector fields at \( 0 \in \mathbb{R}^n \) act on \( S(M) \); the elements of \( W_n \) are interpreted as \( \infty \)-jets of one-parameter families of diffeomorphisms \( \mathbb{R}^n \rightarrow \mathbb{R}^n \), that is, tangent vectors to \( S(\mathbb{R}^n) \) at the point corresponding to the identity mapping \( \mathbb{R}^n \rightarrow \mathbb{R}^n \). Moreover, this interpretation enables us to determine an isomorphism of the space \( W_n \) onto this tangent space. Since a point of \( S(M) \) is an \( \infty \)-jet of the mapping \( \mathbb{R}^n \rightarrow M \), it also determines a mapping \( S(\mathbb{R}^n) \rightarrow S(M) \) which induces an isomorphism of the tangent spaces. Thus, all tangent spaces of the manifold \( S(M) \) may be canonically identified with the tangent space to \( S(\mathbb{R}^n) \), that is, with \( W_n \). This also determines a homomorphism \( W_n \rightarrow \mathfrak{g} (S(M)) \) (its permutability with commutation may be verified trivially), that is, the action of \( W_n \) on \( S(M) \).

By Proposition 1.1, \( S(M) \) is a principal homogeneous \( \mathfrak{g} \)-space.

4. We return to the general theory developed in Sec. 1. Let \( A, S, \mathfrak{g} (S) \) have the same meaning as in Sec. 1, and let \( B \) be a nuclear space. By an exterior differential form of degree \( q \) on \( S \) with values in \( B \) we mean a continuous \( A \)-linear mapping \( A \otimes \mathfrak{g} (S) \rightarrow A \otimes B \) (here \( \Lambda^q \mathfrak{g} (S) \) is the skew-symmetrized \( q \)-th power of the \( A \)-module \( \mathfrak{g} (S) \) relative to tensor multiplication over \( A \)). Let \( \Omega^\cdot = \Omega^0 (S, B) \) denote the space of these forms and let \( \Omega^* = \bigoplus \Omega^q \). Exterior differentiation \( d : \Omega^q \rightarrow \Omega^{q+1} \) is defined by the usual formula:

\[
(d\alpha) (D_1, \ldots, D_{q+1}) = \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \alpha ([D_i, D_j], D_1, \ldots, \hat{D}_i, \ldots, \hat{D}_j, \ldots, D_{q+1}) + \sum_{1 \leq i < q+1} (-1)^i D_i \alpha (D_1, \ldots, \hat{D}_i, \ldots, D_{q+1}).
\]

It is clear from the \( A \)-linearity of the differential form \( \Lambda^q \mathfrak{g} (S) \rightarrow A \otimes B \), that, for every \( s \in S \) it induces a linear (over \( \mathbb{C} \)) mapping \( \Lambda^q_s : S \rightarrow B \) (this accounts for the name).

The complex \( (\Omega^* (S, B), d) \) is called the de Rham complex of the manifold \( S \) with coefficients in \( B \).

We assume now that \( S \) is a principal homogeneous \( \mathfrak{g} \)-space. Then the mapping \( \omega : \mathfrak{g} (S) \rightarrow A \otimes \mathfrak{g} \), inverse to \( \text{id} \otimes \varphi \) (see Sec. 1) is a 1-form on \( S \) with values in \( \mathfrak{g} \).

**PROPOSITION 1.2.** (The Maurer-Cartan formula).

\[
\frac{d\omega}{\omega} = -\frac{1}{2} [\omega, \omega]
\]

that is, \( d\omega (D_1, D_2) = -\frac{1}{2} [\omega (D_1), \omega (D_2)] \) for any \( D_1, D_2 \in \mathfrak{g} (S) \). (The commutator on the right-hand side of the last equality is taken in \( A \otimes \mathfrak{g} \); it is defined by the formula \( [a_1 \otimes g_1, a_2 \otimes g_2] = a_1 a_2 \otimes [g_1, g_2] \).)