CHARACTERISTIC CLASS ENTERING IN QUANTIZATION CONDITIONS

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Recently V. P. Maslov gave a mathematically rigorous treatment of multidimensional asymptotic methods of "quasiclassical" type in the large, i.e., for any number of conjugate points [1, 2]. It turned out that there appeared in the asymptotic formulas certain integers, reflecting homological properties of curves on surfaces of the phase space and closely related to the Morse indexes of the corresponding variational problems. In particular, Maslov defined a one-dimensional class of integer-valued cohomologies whose values on the basis cycles enter into the so-called "quantization conditions."

In this note we give new formulas for the calculation of this class of cohomologies. This class is characteristic in the category of real vector bundles, whose complexification is trivial and trivialized, and also in certain wider categories.

§ 1. NOTATION

1.1. Phase Space

Phase space will be 2n-dimensional real arithmetic space

$$\mathbf{R}^{2n} = \{x\}, \quad x = q,\ p; \quad q = q_1, \ldots, q_n; \quad p = p_1, \ldots, p_n.$$

In $\mathbf{R}^{2n}$ we shall consider the following three structures:

1. The Euclidean structure, given by the scalar quadratic

$$(x,\ x) = p^2 + q^2.$$

2. The complex structure, given by the operator

$$I\colon\ \mathbf{R}^{2n} \to \mathbf{R}^{2n},\ I(p,\ q) = (-q,\ p); \quad z = p + iq, \quad \mathbf{C}^n = \{z\}.$$

3. The simplectic structure, given by the skew-scalar product

$$[x,\ y] = (Ix,\ y) = -[y,\ x]. \tag{1}$$

The automorphism groups of $\mathbf{R}^{2n}$ preserving these structures are called the orthogonal group $\mathbf{O}(2n)$, the complex linear group $\mathbf{GL}(n, \mathbf{C})$, and the simplectic group $\mathbf{Sp}(n)$, respectively. From (1) there follows

LEMMA 1.1. An automorphism preserving two of these structures preserves the third also, so that

$$\mathbf{O}(2n) \cap \mathbf{GL}(n,\ \mathbf{C}) = \mathbf{GL}(n,\ \mathbf{C}) \cap \mathbf{Sp}(n) = \mathbf{Sp}(n) \cap \mathbf{O}(2n) = \mathbf{U}(n).$$

The automorphisms preserving two (and thus all three) structures form the unitary group $\mathbf{U}(n)$. The determinant det of a unitary automorphism is a complex number with modulus 1. Thus there arises a mapping of $\mathbf{U}(n)$ onto the circle

$$\mathbf{SU}(n) \to \mathbf{U}(n) \xrightarrow{\det} \mathbf{S}^1, \tag{2}$$

which is obviously a fibering (the fiber is the group $\mathbf{SU}(n)$ of unitary automorphisms with determinant 1).

1.2. The Lagrangian Grassmanian $\Lambda(n)$

We consider an n-dimensional plane $\mathbf{R}^n \subset \mathbf{R}^{2n}$. It is called Lagrangian if the skew-scalar product of any two vectors of $\mathbf{R}^n$ equals zero. For example, the planes $p = 0$ and $q = 0$ are Lagrangian.*

* The name comes from the "Lagrange brackets" in classical mechanics.

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The manifold of all (nonoriented) Lagrangian subspaces of $\mathbf{R}^{2n}$ is called the Lagrangian Grassmanian $\Lambda(n)$.

From the complex point of view Lagrangian planes can be called real-similar, since there holds

LEMMA 1.2. The unitary group $\mathbf{U}(n)$ acts on $\Lambda(n)$ transitively with stationary subgroup $\mathbf{O}(n)$.

Proof. Let λ be a Lagrangian plane. By (1) this means that the plane $I\lambda$ is orthogonal to λ. Let $\lambda' \in \Lambda(n)$ and ξ, ξ' be orthogonal frames in λ, λ'. Then the automorphism of $\mathbf{R}^{2n}$ carrying ξ into ξ' and $I\xi$ into $I\xi'$ is unitary.

From this lemma it follows that $\Lambda(n)$ is a manifold, $\Lambda(n) = \mathbf{U}(n)/\mathbf{O}(n)$; thus there is a fibering

$$\mathbf{O}(n) \to \mathbf{U}(n) \to \Lambda(n). \tag{3}$$

1.3 The Mapping Det^2: $\Lambda(n) \to S^1$

The determinant of an orthogonal automorphism $A \in \mathbf{O}(n) \subset \mathbf{U}(n)$ equals ± 1. Therefore the square of the determinant of a unitary automorphism carrying the plane $p = 0$ into the Lagrangian plane λ depends only on λ. In this way a mapping is constructed

$$\mathrm{Det}^2\colon \Lambda(n) \to S^1.$$

Denote by $S\Lambda(n)$ the set of Lagrangian planes $\lambda \in \Lambda(n)$ with $\mathrm{Det}^2 \lambda = 1$. On this set the group $\mathbf{SU}(n)$ of unitary unimodular automorphisms acts transitively, and the stationary subgroup of any point is isomorphic to the rotation group $\mathbf{SO}(n)$. Therefore $S\Lambda(n) = \mathbf{SU}(n)/\mathbf{SO}(n)$ is a manifold.

Thus we obtain a diagram (obviously commutative) of six fiberings:

$$\begin{array}{ccccc}
\mathbf{SO}(n) & \to & \mathbf{O}(n) & \xrightarrow{\det} & S^0, \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{SU}(n) & \to & \mathbf{U}(n) & \xrightarrow{\det} & S^1, \\
\downarrow & & \downarrow & & \downarrow z^2 \\
S\Lambda(n) & \to & \Lambda(n) & \xrightarrow{\mathrm{Det}^2} & S^1,
\end{array}$$

where z^2 is the mapping of the circle ($z = e^{i\varphi} \to e^{2i\varphi} = z^2$).

1.4. The Cohomology Class $\alpha \in H^1(\Lambda(n), Z)$

LEMMA 1.4.1. The fundamental group $\Lambda(n)$ is free cyclic,

$$\pi_1(\Lambda(n)) = \mathbf{Z},$$

and its generator goes into the generator of S^1 under the mapping induced by Det^2.

The proof is obtained from the exact homotopy sequences of the left column and lower row of the diagram of section 1.3.

COROLLARY 1.4.2. The one-dimensional homology and cohomology groups of $\Lambda(n)$ are free cyclic:

$$H_1(\Lambda(n), \mathbf{Z}) \simeq H^1(\Lambda(n), \mathbf{Z}) \simeq \pi_1(\Lambda(n)) \simeq \mathbf{Z}.$$

For the generator α of the cohomology group $H^1(\Lambda(n), Z)$ we take the number of rotations of Det^2, i.e., the cocycle whose value on a closed curve $\gamma\colon S^1 \to \Lambda(n)$ is equal to the degree of the composition

$$S^1 \xrightarrow{\gamma} \Lambda(n) \xrightarrow{\mathrm{Det}^2} S^1.$$

(Here S^1 is the circle $e^{i\varphi}$, oriented on the side of increasing φ.)

Example 1.4.3. Let λ be a Lagrangian plane: $\lambda \in \Lambda(n)$. Consider the automorphisms $e^{i\varphi}E \in \mathbf{U}(n)$. The Lagrangian planes $e^{i\varphi}\lambda$ $(0 \le \varphi \le \pi)$ form a closed curve $\gamma\colon S^1 \to \Lambda(n)$, since $e^{i\pi}E = -E$.

The value of the class α on the curve γ equals n.

Indeed, $\det(e^{i\varphi}E) = e^{in\varphi}$, therefore $\mathrm{Det}^2 e^{i\varphi}\lambda = e^{2in\varphi}\mathrm{Det}^2\lambda$.