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LITERATURE CITED

SPECTRAL ASYMPTOTICS OF NON-SELF-ADJOINT ELLIPTIC SYSTEMS OF DIFFERENTIAL OPERATORS IN BOUNDED DOMAINS

K. Kh. Boimatov and A. G. Kostyuchenko

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1. Let \( \Omega \subset \mathbb{R}^n \) be a bounded region satisfying the cone condition. Consider the differential operator

\[
A_0 = A(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \quad D(A_0) = C_0^\infty(\Omega),
\]

with coefficients \( a_\alpha(x) \in C^\infty(\overline{\Omega}; \text{End } \mathbb{C}^l) \) if \(|\alpha| \leq 2m\).

Let the eigenvalues of the symbol

\[
P(x, z) = \sum_{|\alpha| = m} a_\alpha(x) z^\alpha, \quad x \in \Omega, \quad 0 \neq z \in \mathbb{C}^l,
\]

lie on the positive semi-axis \( \mathbb{R}^+ \) and outside an angle \( \Phi = \{ z : |\arg z| < \varphi \}, \varphi \in (0, \pi). \) Consider in \( H = L_2(\Omega)^l \) a closed extension \( A \) of the operator \( A_0 \) satisfying the following conditions:

a) for \( k = [n/(2m)] + 1, \)

\[
D(A^k) \subset W_2^{2mk}(\Omega)^l;
\]

b) for any closed angle \( Q \subset \Phi \) not containing the positive semi-axis there can be found a number \( a > 0 \) such that for \(|z| > a, z \in Q\), the operator \( A - zI \) has its continuous inverse;

c) there exist numbers \( \varphi \equiv (0, \varphi), b > 0 \) such that if \(|z| > b, \arg z = \pm \varphi\), then the inequality

\[
|z - zE|^{-1} \leq M|z|^{-1}
\]

holds.

Mathematical Institute with Computation Center, Academy of Sciences of the Tadzhik SSR.
Throughout the paper (except Secs. 4 and 5) we assume that
\[ n/2m \neq 1, 2, 3, \ldots \]  
(3)

The conditions listed above being met, the operator \( A \) has a discrete spectrum. This follows from (2) and the compactness of the embedding \( W^{2m} \subset L^2 \Omega \).

We denote by \( \lambda_1, \lambda_2, \ldots \) the sequence of eigenvalues of the operator \( A \) in the angle \( \phi \) numbered in nondecreasing order of their moduli with multiplicity taken into account. By \( N(t, x, s) \), we denote the number of the eigenvalues of the matrix \( P(x, s) \) lying on the positive half-axis and on the left of a point \( t \).

There holds the following

**THEOREM 1.** Let the conditions formulated above be met. Then the asymptotic formula
\[ N(t) \sim (2\pi)^{-n} \int_{\Omega} N(t, x, s) \, dx \, ds \]
holds for the function \( N(t) = \sum_{\text{Re} \lambda_j < t} 1 \). At that,

**Remark 1.** Since \( N(t, x, s) = N(1, x, t^{-1/(2m)}) \), we have \( \arg \lambda_j \to 0 \) \((j \to +\infty)\).

\[ N(t) \sim c\pi^{-(2m)} \left( \int_{\Omega} N(1, x, s) \, dx \, ds \right)^{-1/2} \]

2. A closed extension \( A \) satisfying the hypotheses of Theorem 1 can be described with the aid of boundary conditions
\[ B_j u |_{\partial \Omega} = B_j (x, D_x) u (x) |_{\partial \Omega} = 0, \quad j = 1, 2, \ldots, m, \]
if the boundary-value problem
\[ (A (x, D_x) - \zeta I) u (x, z) = f (x, z), \]
\[ B_j (x', D_x) u (x', z) = g_j (x', z), \quad x' \in \Omega, \quad j = 1, 2, \ldots, m, \]
is analytic \([1, 2]\) with a parameter \( z \in \Gamma_\zeta = \{z : \arg z = \zeta\} \) for any \( \zeta \in (0, \pi] \).

Here \( B_j \) are rows of length \( 2 \) consisting of differential operators with respect to \( x \) having complex \( C^\infty \)-coefficients. It is assumed that \( \ord B_j < 2m, \partial \Omega \in C^\infty \).

Note that Theorem 1 is of interest also for the symmetric operators \( A_0 \geq 0 \). In this case it gives the leading term as the spectral asymptotics of non-self-adjoint extensions of the operator \( A_0 \).

3. Let \( A_0 \) be an operator like in Sec. 1, and let all the eigenvalues of symbol (1) lie outside an angle \( \Phi = \{z : \arg z < \Phi\}, \Phi \in (0, \pi) \). Let \( A \) be a closed extension of the operator \( A_0 \) for which condition a) of Sec. 1 holds.

Let the following conditions hold:

(i) there exists a positive number \( \Psi < \min \{\Phi, \pi/(2k)\} (k = [n/(2m)] + 1) \) such that for \( z \) sufficiently large in modulus and satisfying \( \Psi < |\arg z| < \Phi \) the operator \( A - \zeta I \) has its continuous inverse;

(ii) there exists a number \( \Psi_0 \equiv (\Psi, \Phi) \) such that for \( z \) sufficiently large in modulus and satisfying \( \arg z = \pm \Psi_0 \) the inequality
\[ |(A - \zeta I)^{-1}| \leq M |z|^{-1} \]
holds.

We denote by \( N(t) \) the number of the eigenvalues of the operator \( A \) lying in the angle \( \Phi \) and not exceeding \( t \) in modulus.

There holds the following

**THEOREM 2.** The inequality
\[ N(t) \leq Mt^2 + M^{(n-1)/(2m)} \ln t, \]
where \( \rho = [n/(2m)], \theta = 1/2 \), holds for \( t \to 1 \). If, at that,
\[ D ((A^\theta)^k) \subset W^{2m+1} (\Omega), \]
(4)