CONGRUENCES OF FINITE AUTOMATA

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In the analysis of discrete devices representable by the "finite automaton" (sequential machine) model [1] it is necessary to construct simpler models than those obtained by conventional methods. A given finite automaton can be simplified, in particular, by consolidating its states into classes and treating the latter as states of a new automaton. The properties of such classes are investigated below.

1. Weak Congruences and Congruences of a Finite Automaton

We consider finite automata without output $A = (X, S, \delta)$, where $X$ is an input alphabet, $S$ is an alphabet of states, and $\delta$ is an everywhere-defined function of transitions $S \times X \rightarrow S$. Let a certain partition $P$ of set $S$ exist. We associate with this partition the function $\delta^P: S \times X \rightarrow S$, $\delta^P(s, x) = sx^P$, where $sx^P$ is the first state of the chain $s$, $sx$, $sx^2$, ... that belongs, unlike $s$, to another class of the partition $P$; if such a state does not exist, then $\delta^P(s, x) = sx$. As is customary, $sx^0 = s$, $sx^1 = \delta(s, x)$, $sx^2 = \delta(\delta(s, x), x)$, ... For every $x \in X$ the function $\delta^P$ specifies a mapping $D^P_x: S \rightarrow S$.

Definition 1. A partition $K$ of set $S$ is called a weak congruence (w-congruence) if for any $k_i \in K$, $x \in X$: $D^P_x[k_i] \subseteq k_i$, $k_i \in K$. A w-congruence is called a congruence if $\delta^K(s, x) = sx$ for all pairs $(s, x)$.

It is seen at once that the classes of a w-congruence $K$ can be taken as the states of a new automaton without output $A/K = (X, K, \delta^K)$, where $\delta^K(k_i, x) = k_i$, $D^P_x[k_i] \subseteq k_i$, which is logically called the factor automaton of automaton $A$ with respect to w-congruence $K$. Such factor automata can be used as simpler models than $A$ of an object device.

The notion of congruence and the factor automata formed on the basis thereof have been investigated in [2], in which congruences are called admissible partitions. We also note that the classes of a congruence consist of equivalent states [1], whereas the classes of a w-congruence consist of weakly equivalent states [3] for the corresponding automata with output.

Example 1. A finite automaton is given (Fig. 1) with the following w-congruences: $I = \{\{0, 1, 2, 3\}\}$, $K_1 = \{\{3\}, \{0, 1, 2\}\}$, $K_2 = \{\{1\}, \{0, 2, 3\}\}$, $K_3 = \{\{0\}, \{1\}, \{2, 3\}\}$, $K_4 = \{\{0\}, \{1\}\}$, $K_5 = \{\{2\}, \{3\}\}$, $K_6 = \{\{0\}, \{1\}\}$, $K_7 = \{\{2\}, \{3\}\}$, $0 = \{\{0\}, \{1\}\}$. Here the w-congruences $I$, $K_2$, $K_3$, $K_4$, $K_5$, $K_6$, and $0$ are also congruences.

Any automaton has two trivial w-congruences: $0$ consists of single-element classes; $I$ comprises one class. There are also automata that have an arbitrary number of states and only trivial w-congruences. An example of such an automaton is one for which $S = \{1, 2, ..., n\}$, $X = \{x_1^0, x_1^1, x_1^2, ..., x_{n-1}^0, x_{n-1}^1\}$. For each pair $(i, j)$, $i \in S$, $j \in S$ the partial automaton for the input letters $x_{ij}$, $x_{ij}$ is shown in Fig. 2. States $i$ and $j$ can either enter together into a particular class of a w-congruence or enter separately into single-element classes. Since this statement is true for any pair of states, the only possible w-congruences for this automaton are $0$ and $I$. On the other hand, there exists an $n$-state automaton for which any partition of the set of states is a w-congruence (and even a congruence). The partial graph of this automaton for arbitrary $x \in X$ is shown in Fig. 3.

2. Algebra of W-Congruences

It is well known that the operations of intersection and union of partitions can be defined on the set of partitions of an arbitrary set, transforming the set of partitions into a lattice [1, 4]. We now translate these operations to the w-congruences of an automaton.

THEOREM 1. The set of w-congruences of a finite automaton is closed under the union operation $\cup$.

Proof. Let $K_a$ and $K_b$ be two w-congruences of the automaton, and let $K_a = K_e \cup K_b$. The corresponding functions $\delta^K_a$ and mappings $D^K_x$ are denoted by $\delta^a$, $\delta^b$, $\delta^c$, $D^K_a$, $D^K_b$, $D^K_x$. We pick an arbitrary class $k_c \in K_c$. Then

We denote $L = \{k_1^a, k_2^a, \ldots, k_n^a, k_1^b, k_2^b, \ldots, k_m^b\}$. By the definition of the operation of unions we can form a sequence $k_1, k_2, \ldots, k_p$ such that $k_1 \in L$, $k_{i+1} = k_i \cup k_i^p$, $k_i \cap k_i^p \neq \emptyset$, $k_i^p \in L$, $w \in \{a, b\}$ for all $i \leq p$, $k_p = k_0^c$. For any subsets of the set of states $S' \subseteq S$, $S'' \subseteq S$ the relation $D_k[S' \cup S''] = D_k[S'] \cup D_k[S'']$ holds, and if $S' \cap S'' \neq \emptyset$ then $D_k[S'] \cap D_k[S''] \neq \emptyset$. We take an arbitrary input letter $x$ and examine two possible cases.

1) For any state $s \in k_0^c$ the inclusion $\delta^c(s, x) \subseteq k_0^c$ holds. Then $D_k[k_0^c] = k_0^c$ by stipulation.

2) Let the class $k_0^c$ contain a state $s_0$ such that $\delta^c(s_0, x) \notin k_0^c$. Then an $s_u \in k_0^c$ exists such that $s_u x = \delta^c(s_u, x) = \delta^c(s_0, x)$. For the role of $k_0^c$ in the sequence of sets of $L$ we take the class $k_0^d$ to which $s_u$ belongs. Inasmuch as $k_0^d$ is a class of the $w$-congruence $K_0$ and $s_u x \notin k_0^c$, it follows that $D_k[k_0^d] = D_k[k_0^c] \subseteq k_0^d$, $k_0^d \neq k_0^c$, $k_0 \in K_0$. Let $D_k[k_0^d] \subseteq k_0^c$. We consider $k_0^d = k_1 \cup k_1^p$, $D_k[k_0^d] = D_k[k_1] \cup D_k[k_1^p]$. The relation $D_k[k_1] \subseteq k_0^c$ implies $D_k^w[k_1 \cap k_1^p] = k_1^p$ and $D_k^w[k_1^p \cap k_1^p] = k_1^p \cup k_0$, $w \in \{a, b\}$. Since $K_0(k_0^d)$ is a $w$-congruence and a subpartition of $K_0$, we have $D_k^w[k_1^p \cap k_1^p] = k_1^p$ and $D_k^w[k_1^p \cap k_1^p] = k_1^p$. Then by induction we obtain $D_k[k_0^d] = k_0^c$. In every case, therefore, the classes of the partition $K_0$ satisfy Definition 1, i.e., are classes of a $w$-congruence.

This proves the theorem.

It has been shown [2] that the set of congruences is also closed under the intersection operation. However, for $w$-congruences in the general case this is not true, as the example of Fig. 4 demonstrates. Here $K_1 = \{\{1, 2, 3\}, \{4, 5\}\}$ and $K_2 = \{\{1, 2, 4\}, \{3, 5\}\}$ are $w$-congruences. However, $K_1 \cap K_2 = \{\{1, 2, 3\}, \{4, 5\}\}$ is not a $w$-congruence. In the general case, therefore, we can form the algebra $A_0 = (K^*, \cup)$ of all $w$-congruences of a finite automaton with the one operation $\cup$, specified on the set of $w$-congruences $K^*$. This algebra has as its zero the congruence $I(I \cup K = I)$ and as its unit element the congruence $0(0 \cup K = K)$, and it is a semilattice [4], i.e., a semigroup, all elements of which are idempotent.

Next we consider the problem of the basis of algebra $A_0$. The set of $w$-congruences is finite, and so the algebra has a system of generators and a basis that can be used to specify all $w$-congruences of the finite automaton. Inasmuch as $A_0$ is a semilattice, it has a unique basis, which consists of all nondecomposable elements, i.e., elements that cannot be represented by means of the operation $\cup$ in terms of other elements [5]. Thus, the basis of $A_0$ can be found by testing all elements of the algebra for nondecomposability. A substantial reduction of this sequential search process is possible.

We denote by $S^*$ the set of all nonsinglet subsets of the set of states $S$. We take an arbitrary subset $\{S_1, S_2, \ldots, S_k\}$ of $S^*$ satisfying the condition $S_i \cap S_j = \emptyset$ for $i \neq j$. Such a subset and the partition consisting of classes $S_1, S_2, \ldots, S_k$ and all other singlet classes uniquely determine one another. Below, we specify the partition and use it to denote nonsinglet classes.

**Definition 2.** The $w$-congruence determined by the set of nonsinglet classes $\{S_1, S_2, \ldots, S_k\}$ is called dead-ended for $S_i$ if any subset of this set of classes, containing $S_i$, does not specify a $w$-congruence.

Different sets of states of $S^*$ can have more than one dead-ended $w$-congruence, have a unique dead-ended $w$-congruence, or not have any dead-ended $w$-congruences. Clearly, the set of dead-ended $w$-congruences for all sets of states of $S^*$ forms the system of generators for algebra $A_0$. Hence we infer that to construct the basis of $A_0$ it is sufficient to test only dead-ended $w$-congruences for nondecomposability.

Let there be an algebra of $w$-congruences $A_0 = (K^*, \cup)$. We denote by $T(S_i)$ the set of dead-ended $w$-congruences for class $S_i$, and by $T_k(S_i)$ the subset of $T(S_i)$ containing all $w$-congruences having exactly $k$ nonsinglet classes. The set $T(S_i)$ can be constructed by the following procedure.