CONSTRUCTION OF NUMBERING FUNCTIONALS ON COMBINATORIAL SETS

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Numbering and denumbering algorithms for certain combinatorial sets were proposed in [1-5]. The development of these algorithms naturally raises the question of constructing a general numbering method applicable to various combinatorial sets. This question is closely related to the problem of describing combinatorial sets within some general framework, so that specific combinatorial sets are obtained as particular cases of the general description. The concept of r-sample, i.e., an element of the direct product of r sets, was advanced in [6, 7] as a convenient tool for this general description. This concept can be generalized as follows:

Definition 1. An r-sample is an element of the direct product

\[ U = X_0 \times X_1 \times X_2 \times \cdots \times X_r \times X_{r+1}, \]

where the constituent sets \( X_i \) are nonempty and finite and the set \( X_i \) from which the i-th component of the r-sample is drawn is determined by the previously selected components \( x_1, \ldots, x_{i-1} \). The sets \( X_0 \) and \( X_{r+1} \) (which we call the bordering sets) each contain a single element.

We introduce the following conventions:

1) If the constituent sets are number sets, the elements of \( X_0 \) and \( X_{r+1} \) are set equal to zero and are omitted from the r-sample;
2) everywhere, unless otherwise specified, \( i = 1, r \);
3) all quantities with nonpositive subscripts are assumed equal to zero;
4) if the upper limit in summation is less than the lower limit, the sum is assumed to be equal to zero.

These conventions ensure relatively compact formulations in the sequel.

Simple numbering and denumbering procedures for r-samples may be constructed if we fix some numbering of the elements in each of the sets \( X_i \) and identify each element with its number, i.e., for a set \( X_i \) with \( n_i \) elements we take

\[ x_i = \{0, 1, 2, \ldots, n_i - 1\}. \tag{1} \]

The dependence of \( X_i \) on the values of the components \( x_1, \ldots, x_{i-1} \) is thus reduced to the dependence

\[ n_i = n_i(x_1, \ldots, x_{i-1}). \]

In what follows an r-sample is regarded as an element of the direct product of such sets.

The sets (1) have a natural order of integers. This induces the following lexicographic order on the set \( U \).

Definition 2. We say that the r-sample \( v \in U \) lexicographically precedes the r-sample \( u \in U \) if one of the r inequalities is true:

\[ x_i(v) < x_i(u), \quad x_k(u) = x_k(v), \tag{2} \]

where \( k = 0, 1, \ldots, r-1 \), and \( x_i(u) \) is the i-th component of the r-sample \( u \).

Definition 3. The number \( N(u) \) of the r-sample \( u \) is the number of lexicographically preceding r-samples.

By this definition, the numbering functional assigns to every r-sample the number of preceding r-samples.


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By Definition 2, the total number of r-samples lexicographically preceding some r-sample \( u = (x_1^{(r)}, \ldots, x_r^{(r)}) \) is given by a sum of \( r \) terms, where the \( i \)-th term is the number of r-samples that precede \( u \) in virtue of the \( i \)-th condition in (2). Clearly, the sets of r-samples identified by these conditions are nonintersecting.

The \( i \)-th condition in (2) identifies r-samples that precede \( u \) and have the form \( (x_1^{(r)}, x_2^{(r)}, \ldots, x_i^{(r)}, x_{i+1}^{(r)}, \ldots, x_r^{(r)}) \) where the \( i \)-th component is \( x_i = 0, \ldots, x_i^{(r)} - 1 \). To each of these values correspond \( n_{i+1}(x_1^{(r)}, \ldots, x_i^{(r)}, x_{i+1}^{(r)}) \) values of the \( (i + 1) \)-th component, i.e., a total of

\[
\pi' = \sum_{j=0}^{x_i^{(r)}} n_{i+1}(x_1^{(r)}, \ldots, x_i^{(r)}, j)
\]

values. To each of the \( \pi' \) values, in its turn, corresponds a certain set of values of the \( (i + 2) \)-th component, etc. Thus, the numbering functional for r-samples \( u = (x_1^{(r)}, \ldots, x_r^{(r)}) \) may be defined in the form

\[
N(u) = \sum_{i=1}^{r} \sum_{j=0}^{x_i^{(r)}} \sum_{k=0}^{x_{i+1}^{(r)}} \ldots \sum_{l=0}^{x_{i+l-1}^{(r)}} n_{r+1},
\]

where \( n_{r+1} = 1 \) from Definition 1, and the summation limits are given by

\[
v_{ik} = \begin{cases} -1 & n_h(x_1^{(r)}, \ldots, x_{i-1}^{(r)}, i, \ldots, i_{k-1}^{(r)}), \\ k = i + 1, r. \end{cases}
\]

By (1) and Definition 2, the last element in the lexicographically ordered set \( U \) is the r-sample whose \( i \)-th component is \( n_1(n_1 - 1, n_2 - 1, \ldots, n_{i-1} - 1) - 1 \). We denote this r-sample by \( \omega \). Then the number of elements in the set of r-samples \( U \) is

\[
T(U) = N(\omega) + 1.
\]

A denumbering algorithm which reconstructs the r-sample \( u \) from its number \( N(u) \) follows from the equality (3). Thus, if the components \( x_1^{(r)}, \ldots, x_{i-1}^{(r)} \) of the sought r-sample have been found, the component \( x_i^{(r)} \), \( i = \overline{1, r} \) can be obtained from the relation

\[
x_i^{(r)} = \min(a, a \in \{a \mid M_i - D_i(a) < 0, a = 0, n_i - 1\}).
\]

The last component \( x_r^{(r)} = M_r \). Here

\[
D_i(a) = \sum_{j=0}^{a} \sum_{l=0}^{x_{i+l-1}^{(r)}} \ldots \sum_{l=0}^{x_{i+l-1}^{(r)}} n_{r+1},
\]

\[
M_i = N(u), M_i+1 = M_i - D_i(x_i - 1),
\]

\( i = \overline{1, r-1} \).

If some values of the \( k \)-th component of the r-sample are forbidden, the number of elements in \( X_k \) changes. Renumbering the allowed components of \( X_k \) in sequence, we obtain a set \( X_k \) of the form (1). We can now apply (3) to number the r-samples satisfying various constraints, which may be interpreted as restrictions on the number of elements in \( X_k \). The formula (3) also may be applied to number unordered collections of objects, if we agree to write the elements, e.g., in the order of nondecreasing numbers assigned to the objects in the constituent sets.

In the light of these observations, we can construct the functional (3) for some particular combinatorial sets.

**Example 1. Numbering of Permutations with Repetition.** The set of permutations of \( r \) elements taken from \( n \) with repetition may be defined as the direct product of \( r \) sets with \( n \) elements each. A natural trivial generalization involves the direct product of \( r \) sets each with a different number of elements \( n_i \). Using the corresponding summation limits from (4), we rewrite (3) in the form

\[
N(u) = \sum_{i=1}^{r} x_i \cdot \left( \prod_{j=i+1}^{r} n_j \right).
\]

The last r-sample in this case is \( \omega = (n_1 - 1, n_2 - 1, \ldots, n_r - 1) \). Therefore, by (5),

\[
T(U) = 1 + \sum_{i=1}^{r} (n_i - 1) \cdot \left( \prod_{j=i+1}^{r} n_j \right) = \prod_{i=1}^{r} n_i.
\]