LONG-WAVE EQUATION WITH FREE BOUNDARIES. I. CONSERVATION LAWS AND SOLUTION

B. A. Kupershmidt and Yu. I. Manin

0. Notations and Results

0.1. We will study the following system of equations describing the two-dimensional motion of an inviscid, incompressible heavy fluid in the long-wave approximation:

\[
\begin{cases}
  u_t + uu_x - u_y \int_0^y u_x \, d\eta + h_x = 0, \\
  h_t + \left( \int_0^y u \, dy \right)_x = 0.
\end{cases}
\]  

Here \(-\infty < x < \infty\) is the horizontal coordinate, \(0 \leq y\) is the vertical coordinate, and \(t\) the time; \(u(x, y, t)\) is the horizontal velocity component at \((x, y)\) at time \(t\); \(h(x, t)\) is the height of the free surface above \((x, 0)\) at time \(t\). The notation \(u_t\) is an abbreviation for \((\partial / \partial t)u(x, y, t)\), and so forth for the other derivatives. Units are chosen so that the gravitational constant and the density are unity. The integral \(\int_0^y u_x \, d\eta\) is obtained from the continuity condition \(u_x + v_y = 0\), where \(v\) is the vertical velocity component satisfying the boundary condition \(v = 0\) at \(y = 0\). These relations permit one to eliminate \(v\) by expressing it in terms of \(u\), \(v(x, y, t) = -\left( \int_0^y u_x \, d\eta \right)^y\). The details of the derivation may be found in [1], which first demonstrated an unexpected property of system (1): the existence of an infinite number of conservation laws.

The celebrated Korteweg-de Vries equations (KdV) was the first example of what is by now a large family of nonlinear equations with infinitely many conservation laws. Most of these equations may be written in Lax form, \(L_t = [L, A]\), where \(L\) and \(A\) are operators (typically differential or integrodifferential) depending on the unknown functions.

Equation (1) has a series of peculiar properties. We do not know a Lax representation for it. The conservation laws, in our interpretation (which differs from the more formal derivation by Benney), are obtained from a nonlinear integral equation involving a parameter (Theorem 0.3). This equation replaces the evolution equation for the scattering data of the KdV. This same equation allows us to obtain Miura's conservation laws for (1) (see [2]). Furthermore, we establish the Hamiltonian character of a reduced system (1) (with the addi-
tional condition \( u_y = 0 \), the commutativity of the integrals of motion, and we find a class of solutions of the reduced system. In the second paper, these results are generalized to the full system \((1)\).

We now precisely formulate the main results of this paper, and establish various notations to be used throughout. Proofs will be given in the succeeding paragraphs. We begin with the fundamental observation of Benney, which we reproduce with proof in view of the latter's simplicity.

0.2. Benney's Lemma. Define the moments \( A_n(x, t) = \int x^n(x, y, t) \, dy, \ n \geq 0 \). Then \((1)\) implies the following equations for these moments:

\[
A_{n,t} + A_{n+1,x} + nA_{n-1} = 0.
\]

Proof. Multiplying the first equation in \((1)\) by \( nx^n-1 \) and rearranging, we easily obtain

\[
\left( u^n \right)_t + \left( u^{n+1} \right)_x + \left( u^p \right)_x + nA_{n-1} = 0.
\]

Now integrate this from \( y = 0 \) to \( y = h \). The first term becomes \( nA_{n-1}A_{0,x} \), the third term \([because of the second equation in \((1)\)]\) becomes \( u^n(h_t + uh_x) \big|_{y=0} \). Combining the first of these terms with \( \int_0^h (u^n) \, dy \), and the second with \( \int_0^h (u^{n+1}) \, dy \), we obtain \( A_{n,t} \) and \( A_{n+1,x} \), respectively.

In Benney's paper it is shown that there are two sequences of polynomials \( H_n \subseteq Q [A_0, \ldots, A_n] \) and \( F_n \subseteq Q [A_0, \ldots, A_{n+1}] \), such that \((2)\) implies local conservation laws of \((1)\) in the form

\[
H_{n,t} + F_{n,x} = 0, \ n \geq 0.
\]

Miura further shows that there are two more sequences of polynomials \( H_n, F_n \subseteq Q [u, A_0, \ldots, A_{n-1}] \), such that \((2)\) implies local conservation laws of the form\(^*\)

\[
H_{n,t} + F_{n,x} + (H_{n,t})_y = 0, \ n \geq 1, \ v = -\int_0^y u_x \, d\eta.
\]

As usual, assuming the solution to decay sufficiently rapidly at infinity, one may obtain from these local conservation laws a series of constants of the motion: \( \int_0^\infty H_n \, dx, \ n \geq 0 \).

The first two sections of this paper are devoted to a new derivation of \((3)\) and \((4)\). The basis of our method is to obtain the generating function for the \( H_n \) as solution of a certain integral equation with parameter.

0.3. THEOREM. Set \( \Phi(\mu) = \sum_{i=0}^\infty (-1)^i A_i \lambda^{-i+1} = \int_0^h (\lambda + u)^i \, dy \). Then there is a unique solution of the equation

\[
\mu(\lambda) + \Phi(\mu(\lambda)) = \lambda,
\]

a) in the class of formal series of the form \( \lambda + Q [A_0, \ldots, A_n, \ldots] \{[\lambda^{-1}]\} \);

b) in the class functions of \( \lambda \) which are of the form \( \lambda + O(\lambda^{-1}) \) and analytic in some neighborhood of \( \infty \) (depending on \( u(x, y, t), h(x, t) \)).

0.4. THEOREM. System \((1)\) implies the following equations for \( \mu(\lambda) \):

\[
\mu_t - \left( \mu^2 + A_0 \right)_x = 0, \quad \mu_t - \left( \mu^2 + A_0 \right)_x = 0, \quad \mu_t - \left( \mu^2 + A_0 \right)_x = 0.
\]

\[
\left[ \partial \mu/\partial \lambda(\mu + u)^{-1} \right]_t - \left[ \mu \partial \mu/\partial \lambda(\mu + u)^{-1} \right]_y - \left[ \partial \mu/\partial \lambda(\mu + u)^{-1} \right]_y u_x \, d\eta, \mu_t = 0.
\]

\*In Miura's table, the coefficients of \( v \) do not agree with \( H_n \) because of misprints and omitted terms.